Precision of Density Estimates from Fixed-Radius Plots Compared to N-Tree Distance Sampling

Veronica C. Lessard, Thomas D. Drummer, and David D. Reed

ABSTRACT. We computed and compared the statistical properties of the estimators for the number of trees/ha (density) for fixed-radius plot and n-tree distance sampling. In forests with random spatial patterns, n-tree distance sampling density estimators are at least as precise as those of plot sampling if the fixed-radius plot size is less than the ratio of \( n - 2 \) and the expected density, where \( n \) is the number of trees included at an \( n \)-tree location. A similar result holds for the clustered forest, where the ratio is multiplied by a factor involving a constant of heterogeneity. If the expected number of trees per plot and the plot sizes are the same for both the random and clustered spatial patterns, the variance of the plot sampling density estimator for the clustered pattern will always be greater than that of the random spatial pattern. For. Sci. 48(1):1–6.

Key Words: Density-adapted sampling, spatial pattern.

N-Tree distance sampling, referred to as “density-adapted” sampling by Jonsson et al. (1992), provides an alternative forest inventory method to the familiar fixed-radius and variable-radius plot sampling methods. To conduct \( n \)-tree distance sampling, measurements are taken on the \( n \) trees nearest the sampling location. The distance from the plot center to the center of the \( n \)th tree forms the radius of a circular plot. Although the number of trees remains fixed for each sample, the distribution of the size of the circular plot varies with the spatial pattern of the trees.

A number of empirical studies have compared estimates obtained by \( n \)-tree distance sampling with those obtained by fixed-radius plot sampling (Jonsson et al. 1992, Lessard et al. 1994, 1995, Lynch and Rusydi 1999). Jonsson et al. (1992) found that bias and precision of \( n \)-tree distance sampling estimates were comparable with those from 10 m fixed-radius plots (about 0.03 ha). Lessard et al. (1994) found that \( n \)-tree distance sampling was cost-competitive with the more traditional point and plot sampling in estimating the number of trees/ha for three forest types. Lessard et al. (1995) found that the basal area estimates were generally unbiased (within 2% of the true values) for point and plot sampling; \( n \)-tree distance sampling generally underestimated the true mean by 2–10% in nearly random and clustered stands and overestimated by less than 4.5% in a uniform plantation. In plantations, basal area estimates obtained by 7-tree distance sampling were similar to those obtained by point and plot sampling. Similar patterns of overestimation and underestimation of density and volume estimates, due to the spatial patterns of the forests, were found in the simulation study by Jonsson et al. (1992). Lynch and Rusydi (1999) used various sampling techniques, including \( n \)-tree sampling for values of \( n \) ranging from 3 to 10, in Indonesian teak plantations to compare the bias and efficiency of density and volume estimates. \( n \)-tree distance sampling underestimated both density and volume in these uniform plantations. Among the \( n \)-tree methods, they determined 3-tree sampling was most efficient for density estimation, while 5-tree sampling was most efficient for volume estimation.

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Acknowledgments: This study was supported by the Michigan Technological University School of Forestry and Wood Products, United States McIntire-Stennis Act funds, and USDA Forest Service, North Central Research Station and USDA NRCS Natural Resources Inventory and Analysis Institute.

Manuscript received July 7, 1999. Accepted October 11, 2000.

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For comparison of estimators here, the Poisson distribution is used to represent forests with random spatial patterns, and the negative binomial distribution is used to model forests with clustered spatial patterns. The negative binomial distribution, in which clusters of trees are randomly distributed and the number of trees per cluster follows a logarithmic distribution, may be derived from at least five different distributional assumptions and is less restrictive than the Neyman Type A and Thomas Series (Thomas 1949, Anscombe 1950, Pielou 1977). The estimators for density and basal area are unbiased for fixed-radius plot sampling (Palley and Horwitz 1961, Oderwald 1981). Once bias-correction factors have been applied, the n-tree distance sampling density estimators are also unbiased in forests with spatial patterns modeled by Poisson or negative binomial distributions (Moore 1954, Thompson 1956, Eberhardt 1967). Although Eberhardt (1967) compares the relative efficiency of density estimates for plot sampling with those for n-tree distance sampling using the ratio of the squared coefficients of variation for the two methods, he does so in a restricted manner by limiting the fixed-radius plot size to contain an average of one less tree than the sample size of the n-tree distance method.

The objective of this study is to compare the variance of the estimators for the population mean density (number of trees/ha) for fixed-radius plot and n-tree distance sampling in forests that have spatial patterns modeled by either Poisson or negative binomial distributions. Efficiency of the two methods is compared in terms of sampling unit sizes and numbers of trees producing equal variances for an equal number of samples. Density precision comparisons are given between inventory methods within spatial patterns and between spatial patterns within inventory methods.

**N-Tree Distance Sampling Density Estimation**

Before n-tree distance sampling is conducted, the number of trees (n) to include in a sample at each location must be chosen. The number of trees remains fixed for all sampling locations. The radius of the circular plot for a sample is defined as the distance from the sample location to the center of the nth closest tree. This method has been shown to produce density estimates that are biased by a constant factor of n/(n - 1) in stands in which the spatial pattern of trees follows a random or clustered pattern (Morisita 1954, Thompson 1956, Eberhardt 1967). The biased estimates are corrected by multiplying the estimate by the ratio (n - 1)/n, whereas the variance estimates are corrected by multiplying by the square of the same ratio. Let \( r_j \) denote the radius (in meters) of the jth n-tree distance plot and \( \hat{\theta}_{nt} \) denote the n-tree distance sampling estimator of \( \theta \), density (trees/ha). Assuming simple random sampling, the estimated number of trees per plot is obtained by taking the ratio of \( n \) to the plot size, \( n / \pi r_j^2 \). The ratio is converted to an unbiased per ha estimate by multiplying it by 10,000 (n - 1)/n (m\(^2\)/ha). The simple random sampling density estimate (trees/ha) for \( t \) plots may be written as:

\[
\hat{\theta}_{nt} = \frac{10,000}{\pi t} \frac{(n - 1)}{n} \sum_{j=1}^{n} \frac{n}{r_j^2}. \tag{1}
\]

Because \( n \) is fixed, the density estimate depends only on the mean inverse plot radius squared, denoted (1/\( r^2 \)).

Assuming simple random sampling, the estimated (sample) variance of the density estimate (trees/ha\(^2\)) may then be written, based on Cochran (1977), as:

\[
\hat{\text{Var}}(\hat{\theta}_{nt}) = \left( \frac{10,000}{\pi} \frac{(n - 1)}{n} \right)^2 \frac{n}{t (t - 1)} \sum_{j=1}^{t} \left( \frac{1}{r_j^2} - \frac{1}{(1/\sum_{j=1}^{t} r_j^2)} \right)^2. \tag{2}
\]

Let \( W = 1/R^2 \), where \( R \) is the random variable denoting the radius of the n-tree circular plot. We have shown that the n-tree distance sampling estimate depends only on the inverse plot radius squared. Assuming simple random sampling with replacement, the expectation and variance of the n-tree distance sampling density estimator based on mathematical expectation theory are:

\[
E(\hat{\theta}_{nt}) = \frac{10,000}{\pi} \frac{(n - 1)}{n} n E(W); \tag{3}
\]

\[
\text{Var}(\hat{\theta}_{nt}) = \left( \frac{10,000}{\pi} \frac{(n - 1)}{n} \right)^2 n^2 \text{Var}(W). \tag{4}
\]

**Fixed-Radius Plot Sampling Density Estimation**

Formulas for estimators used with fixed-radius sampling are given by many sources (e.g., Husch et al. 1993, p. 165, Schreuder et al. 1993, p. 115, Avery and Burkhart 1994, p. 213, Shiver and Borders 1996, p. 37, Reed and Mroz 1997, p. 177). Let \( n_{(pl)j} \) denote the number of trees sampled on the jth plot when using fixed-radius plot sampling. Estimates of density are obtained by multiplying the number of trees tallied on each plot \( n_{(pl)j} \) by the reciprocal of the plot size (1/\( P \)), and averaging over the \( t \) plots. Let \( \hat{\theta}_{pl} \) be the fixed-radius plot sampling estimator of \( \theta \), density (trees/ha). The simple random sampling density estimate using fixed-radius plot sampling may be written as:

\[
\hat{\theta}_{pl} = \frac{1}{P t} \sum_{j=1}^{t} n_{(pl)j}. \tag{5}
\]

Since the plot size is fixed, the estimate depends only on the mean number of trees per plot, \( \bar{n}_{(pl)} \). The estimated (sample) variance of the density estimate, assuming simple random sampling and based on Cochran (1977), may then be written as:

\[
\hat{\text{Var}}(\hat{\theta}_{pl}) = \frac{1}{P^2} \frac{1}{t (t - 1)} \sum_{j=1}^{t} \left( n_{(pl)j} - \bar{n}_{(pl)} \right)^2. \tag{6}
\]
Let the random variable \( (N_{pl}) \) be the number of trees on a plot and \( n_{pl} \), be a realization of that random variable on a single plot. In the above representation we have shown that once the sampling scheme is determined, an estimate of density from fixed-radius plot sampling depends only on the number of trees per plot. Assuming simple random sampling with replacement, the expected value and variance of the density estimator based on mathematical expectation theory are:

\[
E\left( \hat{\theta}_{pl} \right) = \frac{1}{P} E\left( N_{pl} \right), \tag{7}
\]

and

\[
Var\left( \hat{\theta}_{pl} \right) = \frac{1}{P^2} Var\left( N_{pl} \right). \tag{8}
\]

**Precision Comparisons Between Sampling Methods Within Spatial Patterns**

Variances of density estimates from fixed-radius plots are compared to \( n \)-tree distance sampling within forests that have either random or clustered spatial patterns.

**Random Spatial Pattern**

Thompson (1956) and Eberhardt (1967) show that the expectation for the inverse of the squared plot radius in random spatial patterned forests is:

\[
E(W) = \frac{\pi m_{nt}}{(n - 1)}, \tag{9}
\]

where \( m_{nt} \) is the expected number of trees/m². Substitution of (9) into (3) and simplification yields the expected number of trees/ha:

\[
E(\hat{\theta}_{nt}) = 10,000 \ m_{nt}. \tag{10}
\]

The expected values of the density estimators for plot sampling and for \( n \)-tree distance sampling in a forest with a Poisson spatial pattern are unbiased (Palley and Horwitz 1961, Eberhardt 1967, Oderwald 1981), and the right-hand sides of (10) and (7) may be set equal:

\[
10,000 \ast m_{nt} = \frac{1}{P} E(N_{pl}) \tag{11}
\]

Substituting (10) into (11) and simplifying yields:

\[
E(N_{pl}) = E(\hat{\theta}_{nt})P. \tag{12}
\]

Assuming the Poisson spatial distribution of trees for the random case, Thompson (1956) showed that variance of one over the squared \( n \)-tree plot radius may be written:

\[
Var\left( \frac{1}{W} \right) = \frac{\pi^2 m_{nt}^2}{(n - 1)^2 (n - 2)}, \tag{13}
\]

where \( n > 2 \).

The \( Var(\hat{\theta}_{nt}) \) will be less than the \( Var(\hat{\theta}_{pl}) \) when the following inequality based on (4) and (8) holds:

\[
\left( \frac{10,000}{\pi} \frac{(n - 1)}{n} \right)^2 n^2 Var\left( W \right) < \frac{1}{P^2} Var\left( N_{pl} \right). \tag{14}
\]

Substituting (13) into (14), using the fact that \( E(N_{pl}) = Var(N_{pl}) \) when the spatial pattern of the trees follows a Poisson distribution, and simplifying yields:

\[
\frac{(10,000^2 m_{nt}^2)}{(n - 2)} < \frac{1}{P^2} E\left( N_{pl} \right). \tag{15}
\]

Substituting the results of (12) and (10) into (15) and simplifying:

\[
P < \frac{(n - 2)}{E(\hat{\theta}_{nt})} \tag{16}
\]

Thus, the variance of the fixed-radius plot sampling density estimator will be greater than that of \( n \)-tree distance sampling if the plot size (ha) is less than the ratio of the number of trees included in an \( n \)-tree sample minus two and the expected number of trees/ha. If both sides of the inequality are multiplied by \( E(\hat{\theta}_{nt}) \), it may be seen that the break-even point for precision occurs where the expected number of trees per fixed-radius plot is two less than that of \( n \)-tree distance plots.

**Clustered Spatial Pattern**

Several authors have used the negative binomial distribution to model the locations of trees in forests with clustered spatial patterns. Eberhardt (1967) gave the expected value and variance of one over the squared distance to the center of the \( n \)th tree using \( n \)-tree distance sampling. The expected value is the same as that of the random spatial distribution, given in (10). The variance is:

\[
Var\left( W \right) = \left( \frac{\pi m_{nt}}{n - 1} \right)^2 \frac{(k + n - 1)}{(k - 2)} \tag{17}
\]

where \( n > 2 \) and \( k \) is a constant of heterogeneity. As \( k \to \infty \), the distribution approaches the Poisson distribution and as \( k \to 0 \), the distribution becomes more clustered.

Oderwald (1975) showed that for the negative binomial distribution,

\[
E\left( N_{pl} \right) = m_{pl}, \tag{18}
\]

and

\[
Var\left( N_{pl} \right) = \left( m_{pl} + \frac{m_{pl}^2}{k} \right), \tag{19}
\]

where \( m_{pl} \) is the expected number of trees on a fixed-radius plot. The conditions under which \( n \)-tree distance sampling will be more precise than plot sampling for density are determined by setting

\[
Var(\hat{\theta}_{nt}) < Var(\hat{\theta}_{pl})
\]
and substituting information from (4), (8), (17), and (19):

\[
\left( \frac{10,000}{\pi \frac{n}{n}} \right)^2 \left( \frac{m_{pl}^2}{n - 1} \right)^2 \left( \frac{m_{pl}^2}{n - 2} \right) < \frac{1}{p^2} \left( m_{pl} + \frac{m_{pl}^2}{k} \right).
\]  

(20)

Substituting (18), (7), and (10) into (20), using

\[
E(\hat{\theta}_{mr}) = E(\hat{\theta}_{pl}),
\]

and further simplifying leads to:

\[
P < \frac{n - 2}{E(\hat{\theta}_{mr})} \left( \frac{k}{k + 1} \right).
\]  

(21)

Fixed-radius plot sampling will be less precise than \(n\)-tree distance sampling when the plot size (ha) is less than the product of two ratios: (1) the ratio of \(n - 2\) (where \(n\) is the number of trees included in an \(n\)-tree sample) to the expected number of trees/ha; and (2) the ratio of the constant of heterogeneity \(k\) to \((k + 1)\). The precision relationship shown in (21) for the clustered spatial distribution differs from that of the random spatial pattern (16) only in the ratio involving the constant of heterogeneity. The breakeven point for the precision of the two sampling methods occurs where the expected number of trees per fixed-radius plot is two less than that of the \(n\)-tree distance plots.

**Precision Comparisons Between Spatial Patterns Within Sampling Methods**

The effect of spatial pattern on the precision of density estimates will be examined for both fixed-radius plot and \(n\)-tree distance sampling.

**Fixed-Radius Plot Sampling**

Oderwald (1975) showed that for the Poisson distribution the expected number of trees per plot in the random spatial pattern is \(m_{pl}\). Because the distribution is Poisson, the expected variance of the number of trees per plot will also be \(m_{pl}\). This result and (19) are substituted into (8) separately and compared:

\[
\frac{m_{pl}}{p^2} < \frac{1}{p^2} \left( m_{pl} + \frac{m_{pl}^2}{k} \right).
\]  

(22)

If the expected number of trees per plot and plot sizes are the same for both the random and clustered spatial patterns, the variance of the plot sampling density estimator for the clustered pattern will always be greater than for that of the random spatial pattern.

**N-Tree Distance Sampling**

To compare the precision of \(n\)-tree distance sampling density estimates between random and clustered spatial patterns, substitute (13) and (17) into (4) separately. The constant of heterogeneity \(k\) is always positive, and \(n\) is restricted in (13) and (17) to be greater than two:

\[
\frac{\left( E(\hat{\theta}_{mr}) \right)^2}{n - 2} < \frac{\left( E(\hat{\theta}_{mr}) \right)^2}{\left( k + n - 1 \right)}. \quad (23)
\]

When the expected number of trees/ha, \(E(\hat{\theta}_{mr})\), and \(n\) are equal for sampling in both spatial patterns, the variance of the \(n\)-tree distance sampling density estimate will always be greater in a forest with a clustered spatial pattern than in a forest with a random spatial pattern.

**Discussion**

Table 1 summarizes the maximum fixed-radius plot size (ha) for which the variance of the \(n\)-tree density

<table>
<thead>
<tr>
<th>Number of trees on an (n)-tree plot ((n))</th>
<th>The expected number of trees/ha (E(\hat{\theta}_{mr}))</th>
<th>Maximum fixed-radius plot size ((\text{ha}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k=10)</td>
<td>(k=50)</td>
<td>(k=100)</td>
</tr>
<tr>
<td>(7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.091</td>
<td>0.099</td>
</tr>
<tr>
<td>100</td>
<td>0.045</td>
<td>0.049</td>
</tr>
<tr>
<td>150</td>
<td>0.030</td>
<td>0.033</td>
</tr>
<tr>
<td>250</td>
<td>0.018</td>
<td>0.020</td>
</tr>
<tr>
<td>500</td>
<td>0.009</td>
<td>0.010</td>
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<tr>
<td>(5)</td>
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<tr>
<td>50</td>
<td>0.055</td>
<td>0.059</td>
</tr>
<tr>
<td>100</td>
<td>0.027</td>
<td>0.029</td>
</tr>
<tr>
<td>150</td>
<td>0.018</td>
<td>0.020</td>
</tr>
<tr>
<td>250</td>
<td>0.011</td>
<td>0.012</td>
</tr>
<tr>
<td>500</td>
<td>0.005</td>
<td>0.006</td>
</tr>
<tr>
<td>(3)</td>
<td></td>
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</tr>
<tr>
<td>50</td>
<td>0.018</td>
<td>0.020</td>
</tr>
<tr>
<td>100</td>
<td>0.009</td>
<td>0.010</td>
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<tr>
<td>150</td>
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<td>0.004</td>
</tr>
<tr>
<td>500</td>
<td>0.002</td>
<td>0.002</td>
</tr>
</tbody>
</table>

*In stands of random spatial pattern, the comparison is given by \(P < (n - 2)/E(\hat{\theta}_{mr})\), where \(P = \text{fixed-radius plot size (ha)}, \ E(\hat{\theta}_{mr}) = \text{expected number of trees/ha}, \) and \(n\) is the number of trees to be included on an \(n\)-tree distance plot. For clustered stands, modeled by the negative binomial distribution, the comparison is calculated as \(P < (m_{pl}^2)/E(\hat{\theta}_{mr}) \times (k/(k + 0))\). The parameter \(k\) specifies the degree of clustering; as \(k \to \infty\), the spatial pattern becomes more random and as \(k \to 0\), the spatial pattern becomes more clustered.
estimator is expected to be less than that of fixed-radius plot sampling for different densities, numbers of trees included on n-tree plots, and spatial patterns as derived from (16) and (21). Because Lessard et al. (1994) found sampling more than seven trees on an n-tree plot to be impractical in field applications, Table 1 makes comparisons using 7-tree, 5-tree, and 3-tree distance sampling. For random and clustered patterns modeled by the Poisson and negative binomial distributions, respectively, the spatial pattern has a minimal impact on the precision trade-offs relating fixed-radius plot size and the number of trees sampled on an n-tree plot. Density has a major effect on the relative precision of n-tree sampling and plot sampling. As density increases, it takes a smaller plot size to get n trees. By rearranging (16) as:

\[ P E(\hat{\theta}_n) < n - 2. \]

it may be seen that for the random spatial distribution, the break even fixed-radius plot sizes contain an average of two fewer trees than the relevant n-tree sample. Results of Table 1 and the solution of (21) for n = 2 both show that this is approximately true for the clustered forest, as well.

The variance formulas compared here are for the population variance of the density estimates using the different sample units, defined as fixed-area plots or n-tree samples, and spatial patterns modeled by the Poisson and negative binomial distributions. The extent to which the sample variance of the density estimates for the two sampling methods resembles the true variance depends on how closely the actual stands to which these methods are applied follow the assumed spatial patterns (i.e., Poisson or negative binomial distributions). In application, we do not know the true variance and therefore use the estimated (sample) variance of the estimate and the size of the estimated standard error of the resulting estimate. Although the estimated variance of n-tree samples may be larger, one can compensate by taking more samples in order to reduce the standard error of the resulting estimate (Lessard et al. 1994). This implies further trade-offs involving the relative costs of taking the different types of samples.

To illustrate the cost trade-offs, we use an example from Lessard et al. (1994), in which the time needed for field samples using fixed-radius plot, variable-radius point, and n-tree distance sampling relative to the variability of the estimates were compared. Based on that study, it took about four times as long to sample a 0.08-ha fixed-radius plot as it did to take a 5-tree distance sample in a northern hardwoods forest type. Standard errors (i.e., the square root of the ratio of the estimated variance and the respective sample size) may be used to compare efficiencies of sampling methods given a fixed amount of sampling time. Assuming the relative size of the sample variance closely resembles true variance and using the results of (14) for a given time spent sampling, the variance estimates of the n-tree density estimates should be divided by 4 (the number of 5-tree samples measured for every 0.08 ha fixed-radius plot sample in a fixed amount of time). The results simplify to:

\[ E(\hat{\theta}_n) < \frac{4(n - 2)}{P}. \]

In this example application, because n = 5 and P = 0.08 ha, it would be advantageous to use 5-tree distance sampling over 0.08 ha plot sampling if E(\hat{\theta}_n) were less than 150 trees/ha.

For equal probability sampling, fixed-radius plot sampling was shown to be more precise in forests with random spatial patterns than in forests with clustered spatial patterns for density estimation. In forests with random spatial patterns, n-tree distance sampling density estimates are also more precise than in forests of clustered spatial patterns.

**Conclusions**

Relatively simple inequalities define the precision comparisons of fixed-area and n-tree density estimates for particular spatial patterns of tree distribution. The trade-offs for using one method over the other are affected by spatial pattern and density, but spatial pattern has a relatively minor effect on the trade-offs, while expected density has a major effect. Finally, trade-offs in application (standard error) depend on the relative costs of taking plot and n-tree samples in a particular forest type. Although the variance is often greater with n-tree sampling, this can be overcome in at least some cases by increasing the numbers of sample units for a given cost. In an inventory design involving a combination of sampling techniques, n-tree distance sampling could be utilized to make the design more cost effective. For example, while variable-radius point sampling is often used to inventory large trees, small trees are often measured on fixed-radius plots. By using n-tree distance sampling for the small tree portion of the inventory, the number of small trees to be measured on each plot could be designed into the sampling regime to increase efficiency.

**Literature Cited**


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