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Fixed and Equilibrium Endpoint Problems in Uneven-Aged Stand Management

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ABSTRACT. Studies in uneven-aged management have concentrated on the determination of optimal steady-state diameter distribution harvest policies for single and mixed species stands. To find optimal transition harvests for irregular stands, either fixed endpoint or equilibrium endpoint constraints can be imposed after finite transition periods. Penalty function and gradient methods are presented to solve these problems. The methods are demonstrated with a stage-structured model for projecting stands that contain mixtures of California white fir (*Abies concolor* [Gord. & Glend.] Lindl. (Iowiana [Gord.])) and red fir (*Abies magnifica* A. Murr.). With present value as the efficiency criterion, optimal transition strategies are computed for three kinds of target steady states: the extremal steady state associated with an infinite time horizon dynamic optimization problem, an investment-efficient steady state, and a maximum sustainable rent steady state. Harvest regimes that convert to investment-efficient or maximum sustainable rent steady states are dominated by red fir and are suboptimal compared to transition regimes that convert to the extremal steady state, which includes only white fir. The fixed endpoint regimes are compared with transition strategies that are obtained with equilibrium endpoint constraints that do not require a particular steady-state stand structure. Transition regimes that convert to the extremal steady state are suboptimal compared to regimes that solve the more general equilibrium endpoint problem, and the present values of these two kinds of regimes converge as the transition period lengthens. The species composition and structure of the steady states found by solving the equilibrium endpoint problem depend on the transition period length. FOR. SCI. 33(4):908–931.

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MODELS FOR UNEVEN-AGED MANAGEMENT have been developed that allow the determination of the diameter class harvesting rates that maximize the present value of an existing stand over an extended planning horizon (see Haight et al. 1985 and Haight 1985). Optimal management regimes often involved large variations in harvest levels over time, and steady states were achieved only after long transition periods. Stand management objectives that include steady-state harvesting in addition to the maximization of present value can be incorporated into a dynamic optimization model by constraining the harvest level to be in a steady state after a finite number of transition harvests. This paper focuses on issues associated with dynamic harvesting problems that have either fixed endpoint or equilibrium endpoint constraints.

Fixed endpoint problems involve the determination of a target steady state and a transition regime that reaches the target after a finite transition

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period. Many studies have investigated optimal steady-state harvesting independently of transition regimes. Optimal volume harvest levels and cutting cycle lengths have been determined using whole-stand diameter-free models for stand growth (Chang 1981, Hall 1983). Optimal steady-state diameter distributions and harvest rates have been found using whole-stand diameter-class simulators (Adams 1976, Buongiorno and Michie 1980, Martin 1982). The steady states were optimal in the sense that they satisfied an economic stocking criterion that equated the marginal value growth percent of the stand to the discount rate. Since this criterion is equivalent to the land expectation value criterion for determining the rotation ages of even-aged stands (see Chang 1981 and Hall 1983), optimal steady states are called investment-efficient. In contrast to the marginal value growth percent criterion, Rideout (1985) proposed a "managed forest value" criterion, which seeks the steady state that maximizes the present value of an infinite series of harvested volumes. Steady states that satisfy this criterion are called maximum sustainable rent solutions, and they are independent of the discount rate (see Getz 1986). One study has focused on steady-states that are found in the context of a dynamic harvesting problem that seeks the diameter class harvesting rates that maximize the present value of an existing stand over an infinite time horizon. A steady-state solution to an infinite time horizon optimization problem is called an extremal steady state. The extremal steady state differs, in general, from an investment-efficient steady state (Haight 1985), and it is equivalent to a maximum sustainable rent steady state only in the special case where the discount rate is zero (Getz 1986). Using a maximum present value objective, Adams and Ek (1974) formulated and solved a three-period fixed endpoint problem. Computational limitations prevented the authors from examining the present value impacts of longer transition periods, and they did not examine the present value impacts of converting to steady states that were determined with alternative criteria.

Equilibrium endpoint problems involve the determination of transition and steady-state harvest levels with equilibrium endpoint constraints that do not require the achievement of a specific target stand structure. Michie (1985) formulated and solved an equilibrium endpoint problem using a fixed parameter matrix model for stand growth and linear programming methods. The resulting steady states depended on the initial stand structure and the conversion period length, and they approached an investment-efficient stand structure as the conversion period lengthened. Computational limitations prevented the author from solving problems with three or more transition harvests. In addition, linear programming methods cannot be used when stand growth is projected with density dependent nonlinear models.

The purpose of this paper is to formulate and solve finite time horizon problems that have either equilibrium endpoint constraints (e.g., Michie 1985) or fixed endpoint constraints (e.g., Adams and Ek 1974). Numerical solution procedures that involve penalty functions and gradients are presented and demonstrated in the context of a stage-structured model for stands that contain mixtures of California white fir and red fir. The stage-structured model includes relations for regeneration and growth that are density dependent and nonlinear. Numerical solutions to the equilibrium endpoint problem are used to determine the impact of transition period length on the optimal steady-state stand structure and species composition. Solutions to the fixed endpoint problem are used to determine the impacts of converting to extremal, investment-efficient, and maximum sustainable rent steady states on the present value of conversion and steady-state harvesting.

FINITE TIME HORIZON HARVESTING

A General Resource Management Problem

In this section an infinite time horizon resource management problem is formulated, and in the following two sections, the general problem is converted to finite time horizon problems using equilibrium endpoint and fixed endpoint constraints. Consider a population represented by $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))'$, where $x_i(t)$ is the number of individuals in the ith growth stage at the beginning of time period t. For biological realism constrain $\mathbf{x}(t)$ to lie in R^{+n}, the nonnegative quadrant of R^n. Assuming that the population under consideration is a resource system in which individuals can be harvested from each growth stage, define a control vector $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))'$, where $u_i(t)$ is the number of individuals harvested from the ith growth stage at the end of time period t. Since $\mathbf{u}(t)$ represents the levels at which the resource is exploited, $\mathbf{u}(t)$ belongs to a suitably defined set $U \in R^n$, where

$$U = \{\mathbf{u}(t) \in \mathbf{R}^n \mid 0 \leq u_i(t) \leq \bar{u}_i(t), i = 1, \ldots, n\}, \tag{1}$$

and $\bar{u}_i(t)$ is an upper bound equal to the number of individuals in stage i at the end of time period t before harvest. There is usually a practical unit of time associated with the resource system. For tree populations we assume that the unit of time is 5 years, and thus t increases in 5-year intervals. For generality we allow an immediate harvest $\mathbf{u}(0)$ from a population with initial stage $\mathbf{x}(0) = \mathbf{x}_0$ before the dynamics begin. The population dynamics are defined with the general system of nonlinear difference equations

$$\begin{aligned} \mathbf{x}(1) &= \mathbf{x}(0) - \mathbf{u}(0) \\ \mathbf{x}(t+1) &= \mathbf{q}(\mathbf{x}(t),\mathbf{u}(t)),\ t = 1,2, \ldots . \end{aligned} \tag{2}$$

The specific structure of the growth vector $\mathbf{q}$ is described in a later section.

The problem of exploiting a general renewable resource can be formalized as follows. Define $R(\mathbf{x}(t),\mathbf{u}(t))$ as the revenue obtained at the end of time period t, where the resource is in state $\mathbf{x}(t)$ at the beginning of this time period and the harvest control $\mathbf{u}(t)$ takes place at the end of the time period. Let δ denote a discount factor related to the real discount (interest) rate r by $\delta = 1/(1 + r)$. The infinite time horizon management problem can now be defined as:

$$\max_{\{\mathbf{u}(0),\mathbf{u}(1),\mathbf{u}(2),\ldots\}} J(\mathbf{x}_0) = \sum_{t=0}^{\infty} \delta^t R(\mathbf{x}(t),\mathbf{u}(t)) \tag{3}$$

subject to the growth Equation (2) and initial state $\mathbf{x}_0$. Suppose this problem has a solution denoted by $\tilde{\mathbf{u}}(t)$, $t = 0, 1, \ldots$, with corresponding state $\tilde{\mathbf{x}}(t)$, $t = 1, 2, \ldots$, and present value $J^*(\mathbf{x}_0)$.

Since it is impossible to numerically solve an infinite time horizon optimal control problem, for computational purposes a related finite time horizon problem must be formulated. The infinite time horizon problem can be approximated by a finite time horizon problem that includes a transition period with length T and a terminal state $\mathbf{x}(T)$. If the terminal state is free and no cost is attached to leaving the resource in any particular state, the resource system is often heavily exploited in the time intervals leading up to and including T. Noting that the free terminal condition did not affect the optimal solution to the initial period harvest level $\tilde{\mathbf{u}}(0)$ for large T, Haight (1985) avoided this problem with a sequential solution algorithm that repeatedly solved for and implemented the first period harvest control.

An alternative finite time horizon problem formulation involves constraining the resource system to be in equilibrium at the beginning of time period T; that is, $\mathbf{x}(t+1) = \mathbf{x}(t) = \mathbf{x}$, say, for $t = T, T+1, \ldots$. This formulation is appealing when external considerations require the achievement and maintenance of a steady-state flow of products from the renewable resource in addition to the primary objective of maximizing the present value of the resource system. Two methods for achieving a terminal equilibrium state at time T are presented in the following subsections. The first formulation imposes a general equilibrium endpoint constraint. The second formulation imposes a fixed endpoint constraint where the fixed endpoint is a specified steady-state stand structure.

Equilibrium Endpoint Problem

From growth Equation (2) and the equilibrium constraint, it follows that

$$\mathbf{x} = \mathbf{q}(\mathbf{x},\mathbf{u}). \tag{4}$$

Let $\mathbf{x_u}$ denote a solution to Equation (4) for a given $\mathbf{u} \in U^e$ defined by

$$U^e = \{\mathbf{u} \in U \mid (\mathbf{x_u},\mathbf{u}) \text{ is a biologically feasible solution to (4)}\}. \tag{5}$$

The equilibrium endpoint problem is defined as

$$\max_{\{\mathbf{u}(t), t=0,1,\ldots,T-1\}} J_T(\mathbf{x}_0) = \sum_{t=0}^{T-1} \delta^t R(\mathbf{x}(t),\mathbf{u}(t)) + \frac{\delta^T}{1-\delta} R(\mathbf{x_u},\mathbf{u}) \tag{6}$$

where $\mathbf{x_u} = \mathbf{x}(T)$ and $\mathbf{u} \in U^e$. Note that the second term on the right side corresponds to $\Sigma_{t=T}^{\infty} \delta^t R(\mathbf{x_u},\mathbf{u})$. The idea here is to maximize $J_T(\mathbf{x}_0)$ for initial stand condition $\mathbf{x}_0$ subject to the growth dynamics (2) holding for $t = 0, 1, \ldots, T-1$ and the pair $(\mathbf{x_u},\mathbf{u})$ satisfying equilibrium Equation (4). If $J_T^*(\mathbf{x}_0)$ is the value corresponding to the optimal solution to the T-horizon equilibrium endpoint problem, then $J_T^*(\mathbf{x}_0)$ approximates $J^*(\mathbf{x}_0)$, the value of the infinite time horizon problem, and converges to it as $T \to \infty$. The difference between $J_T^*(\mathbf{x}_0)$ and $J^*(\mathbf{x}_0)$ can be regarded as the cost associated with the constraint that the system must be in equilibrium for $t \geqslant T$. It is worth noting that the optimal equilibrium pair $(\mathbf{x_u},\mathbf{u})$ depends on $\mathbf{x}_0$, T, and δ. The structure of problem (6) is equivalent to the linear programming formulation given by Michie (1985) except that the growth dynamics (2) may contain density-dependent nonlinear functions.

Fixed Endpoint Problem

The infinite time horizon problem (3) can also be approximated by a finite time horizon problem in which the terminal endpoint is constrained to be a fixed equilibrium endpoint. One choice for the fixed endpoint is the extremal steady state associated with the infinite time horizon problem (3). Conditions for the extremal steady state can be constructed using Pontryagin's Maximum Principle and the associated *current value* Hamiltonian. Specifically, Pontryagin's Maximum Principle (as modified to include discounting—see Haurie 1982) states that a solution pair $(\tilde{\mathbf{x}}(t),\tilde{\mathbf{u}}(t))$ is an extremal for the above infinite time horizon problem only if there exists a *current value* costate variable $\lambda(t)$ and *current value* Hamiltonian (note that $'$ is used to denote the transpose of a vector)

$$\mathrm{H}(\lambda(t+1),\mathbf{x}(t),\mathbf{u}(t),t) = \lambda'(t+1)\delta\mathbf{q}(\mathbf{x}(t),\mathbf{u}(t)) + R(\mathbf{x}(t),\mathbf{u}(t)) \tag{7}$$

such that for $t = 0, 1, 2, \ldots$,

$$\lambda'(t) = \lambda'(t + 1)\delta \frac{\partial \mathbf{q}}{\partial \mathbf{x}} (\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t)) + \frac{\partial R}{\partial \mathbf{x}} (\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t)) \tag{8}$$

and

$$\lambda'(t + 1)\delta \frac{\partial \mathbf{q}}{\partial \mathbf{u}} (\bar{\mathbf{x}}(t),\mathbf{u}(t)) + \frac{\partial R}{\partial \mathbf{u}} (\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t)) = 0. \tag{9}$$

Note that Equations (8) and (9) only hold for $\bar{\mathbf{u}}(t)$ lying in the interior of the feasible set U defined in (1). If $\bar{\mathbf{u}}(t)$ lies on the boundary of U, then the more general condition

$$\mathrm{H}(\lambda(t + 1),\bar{\mathbf{x}}(t),\bar{\mathbf{u}}(t),t) = \max_{\mathbf{u} \in U} \mathrm{H}(\lambda(t + 1),\bar{\mathbf{x}}(t),\mathbf{u},t) \tag{10}$$

holds.

If a stable extremal steady state exists, it is the solution to Equations (8) and (9) and steady-state condition (4). The extremal steady state depends solely on δ and is denoted $(\mathbf{x}_\delta,\mathbf{u}_\delta)$. Note that, letting λ'_δ be the corresponding equilibrium value for the *current value* costate variable, Equations (8) and (9) at equilibrium reduce to

$$\lambda'_\delta\left(\delta \frac{\partial \mathbf{q}}{\partial \mathbf{x}} (\mathbf{x}_\delta,\mathbf{u}_\delta) - 1\right) + \frac{\partial R}{\partial \mathbf{x}} (\mathbf{x}_\delta,\mathbf{u}_\delta) = 0, \tag{11}$$

and

$$\lambda'_\delta\delta \frac{\partial \mathbf{q}}{\partial \mathbf{u}} (\mathbf{x}_\delta,\mathbf{u}_\delta) + \frac{\partial R}{\partial \mathbf{u}} (\mathbf{x}_\delta,\mathbf{u}_\delta) = 0. \tag{12}$$

Equations (11) and (12) are equivalent to the steady-state conditions for infinite time horizon resource management problems derived by Knapp (1983) and Horwood and Whittle (1986) using dynamic programming.

With an extremal steady state target, the fixed endpoint problem is

$$\max_{\{\mathbf{u}(t),t=0,1,\ldots,T-1\}} I_T(\mathbf{x}_0) = \sum_{t=0}^{T-1} \delta^t R(\mathbf{x}(t),\mathbf{u}(t)) + \frac{\delta^T}{1 - \delta} R(\mathbf{x}_\delta,\mathbf{u}_\delta) \tag{13}$$

subject to the growth dynamics (2) holding for $t = 0, 1, \ldots, T - 1$ and the endpoint constraint $\mathbf{x}(T) = \mathbf{x}_\delta$. If $I_T^*(\mathbf{x}_0)$ is the value corresponding to the optimal solution to the T-horizon fixed endpoint problem, then $I_T^*(\mathbf{x}_0)$ approximates $J^*(\mathbf{x}_0)$, the value of the infinite time horizon problem, and approaches it as $T \rightarrow \infty$. The additional constraints on $\mathbf{x}(T)$ mean that $I_T^*(\mathbf{x}_0)$ is less than or equal to $J_T^*(\mathbf{x}_0)$ for any finite T. Associated with the fixed endpoint problem is the question of reachability of the target set, that is, do any controls exist that drive the system to the specified endpoint $\mathbf{x}(T)$ in the allocated time interval T.

A second choice for the fixed endpoint is an *investment-efficient* steady state, which has been advocated by a generation of forest economists (Duerr and Bond 1952, Adams and Ek 1974, Adams 1976, Buongiorno and Michie 1980, Chang 1980, Martin 1982, Hall 1983). Investment-efficient steady states satisfy an economic stocking criterion that equates the marginal value growth percent of the stand to the discount rate. Investment efficient steady states are determined independently of the transition regime by solving a maximization problem involving the present value of the steady state pair $(\mathbf{x}_\mathbf{u},\mathbf{u})$ (that is, the term $\frac{\delta}{1-\delta}R(\mathbf{x}_\mathbf{u},\mathbf{x})$)) and a term $R(\mathbf{x}_\mathbf{u},\mathbf{x}_\mathbf{u})$ representing the opportunity cost of the residual growing stock. The opportunity cost is the revenue that could be obtained by clearcutting the residual growing stock $\mathbf{x}_u$ (that is, $\mathbf{u} = \mathbf{x}_u$). The maximization problem is

$$\max_{\{\mathbf{u} \in U^e\}} \left[\frac{\delta}{1 - \delta} R(\mathbf{x}_u, \mathbf{u}) - R(\mathbf{x}_u, \mathbf{x}_u) \right]. \tag{14}$$

This formulation has appealed to forest economists because if the growing stock $\mathbf{x}$ is viewed as a capital investment, then problem (14) is equivalent to maximizing land expectation value as defined by the Faustmann formula (Chang 1980, Hall 1983). Denote $(\check{\mathbf{x}}, \check{\mathbf{u}})$ as the optimal solution to problem (14). It turns out that a solution $(\check{\mathbf{x}}, \check{\mathbf{u}})$ satisfies conditions (11) and (12) for the extremal steady state only in the special case where the revenue function R is a linear combination of the harvest level and $\check{\mathbf{u}} > 0$ (see Haight 1985 for details). Thus, for sufficiently large T, a solution to the fixed endpoint problem (13) that is constrained to achieve an investment-efficient steady state has, in most cases, a lower present value relative to the present value of the solution to the fixed endpoint problem that is constrained to achieve the extremal steady state.

A third choice for the fixed endpoint is the steady state that maximizes managed forest value. Managed forest value measures the present value of an infinite series of steady-state harvests (see Rideout 1985). In the context of problem (6) managed forest value is $\frac{\delta}{1-\delta} R(\mathbf{x}_u, \mathbf{u})$, the present value at time T of the steady state pair $(\mathbf{x}_u, \mathbf{u})$. If $(\check{\mathbf{x}}, \check{\mathbf{u}})$ is the steady-state solution to the problem of maximizing managed forest value,

$$\max_{\{\mathbf{u} \in U^e\}} \left[\frac{\delta}{1 - \delta} R(\mathbf{x}_u, \mathbf{u}) \right], \tag{15}$$

then $(\check{\mathbf{x}}, \check{\mathbf{u}})$ is a maximum sustainable rent solution that is independent of the discount factor. Maximum sustainable rent solutions are the same as the extremal steady state to the infinite time horizon problem only in the special case where $\delta = 1$. To see this, note that conditions (11) and (12) for the extremal steady state provide a set of necessary conditions for $(\mathbf{x}, \mathbf{u})$ to maximize $R(\mathbf{x}, \mathbf{u})$ subject to the constraint $\mathbf{x} = \delta \mathbf{q}(\mathbf{x}, \mathbf{u})$; that is, as $\delta \rightarrow 1$, the extremal steady state $(\mathbf{x}_\delta, \mathbf{u}_\delta)$ approaches the maximum sustainable rent solution $(\check{\mathbf{x}}, \check{\mathbf{u}})$ to problem (15). Thus, for $\delta < 1$ and sufficiently large T, a solution to the fixed endpoint problem (13) that is constrained to achieve a maximum sustainable rent steady state has a lower present value relative to the present value of the solution to the fixed endpoint problem that is constrained to reach the extremal steady state.

Discussion

The formulation for determining investment-efficient steady states may include different definitions of the opportunity cost of the residual growing stock. In expression (14) the opportunity cost is calculated as though the residual stand $\mathbf{x}_u$ is clearcut, and this is consistent with the literature on determining investment-efficient steady states. The reviewers pointed out good arguments for different ways to evaluate opportunity cost. One definition evaluates only the merchantable trees since it is not prudent to liquidate the unmerchantable trees (at a cost) when undertaking the alternative investment. The second definition attaches a positive value to unmerchantable trees since they have a positive value in the future as they become merchantable. Regardless of how the opportunity cost is calculated, investment-efficient steady states are not the same as extremal steady states to the infinite time horizon dynamic problem (3). As a result, solutions to the fixed endpoint problem with any investment-efficient steady state as a target are suboptimal compared to solutions that are constrained to reach the extremal steady state.

The fixed and equilibrium endpoint problems are analogous to the forest-level harvest scheduling problems formulated and solved by Nautiyal and Pearse (1967). These authors used linear programming to address the problems of converting a set of even-aged stands with an irregular area versus age distribution to a normal forest, which has a distribution of age classes that provides a steady-state harvest level. The fixed endpoint problem is analogous to the problem of determining the optimal pattern of harvests from an irregular forest during its conversion to normality where the length of the conversion period and the terminal area versus age distribution are given. The equilibrium endpoint problem is analogous to the problem of determining the area versus age distribution of the normal forest that maximizes the present value of the conversion and steady-state harvests for a given conversion period.

While the fixed and equilibrium endpoint problems are analogous to constrained forest-level harvesting problems, the steady-state behavior of the unconstrained uneven-aged management model differs from the steady-state behavior of the unconstrained forest-level model. Under certain conditions (for example, convexity of the revenue and growth functions) there exist stable, single-period steady states for the infinite time horizon uneven-aged management problem (3) (see Horwood and Whittle 1986). In contrast, the linear forest-level harvesting problem does not converge to a single-period steady state characterized by a normal forest. The optimal harvest pattern is to clearcut each age class at the Faustmann-optimal rotation age (see Johannson and Lofgren 1985).

PENALTY FUNCTION METHODS

A Stage-Structured Harvesting Model

The numerical methods for solving the fixed and equilibrium endpoint management problems are discussed in the context of a stage-structured model for projecting growth and yield. In this model, the right hand side of Equation (2) is characterized as a transformation of the state vector $\mathbf{x}(t) = (\mathrm{x}_1(t), \ldots, x_n(t))'$ using an n-dimensional matrix model. Each element of the state vector represents the number of trees in a specified growth stage at the beginning of time period t. Growth stages include height classes for saplings less than 6.0 ft in height and diameter (at breast height) classes for trees taller than 6.0 ft. For management analysis trees in each diameter class are also assigned an average height, crown attributes, and stem volume.

The elements of the transition and survival matrices and the regeneration vector are nonlinear functions that depend on k aggregations of the elements of the state vector $\mathbf{x}(t)$. Each aggregate variable represents a measure of stand density that is defined as a weighted sum of the number of trees in each stage. The aggregate density variables are denoted by the vector $\mathbf{y}(t) = (\mathrm{y}_1(t), \ldots, y_k(t))'$ and are defined by the transformation

$$\mathbf{y}(t) = A\mathbf{x}(t) \tag{16}$$

where A is a $k \times n$ matrix with elements a_{ij}, $i = 1, \ldots, k, j = 1, \ldots, n$.

The elements of the survival and transition matrices are composed of the scalar functions $s_i(\mathrm{y}(t))$ and $p_i(\mathrm{y}(t))$, taking values on the interval [0,1], and denoting the proportion of trees in stage i at time t that respectively survive the time period $(t, t + 1]$, and move into the next stage at time $t + 1$. Note that the number of trees in stage i at time t that remain in stage i at time $t + 1$ is then given by $(1 - p_i(\mathrm{y}(t)))$. Define $n \times n$ matrices $S(\mathrm{y}(t))$ and $P(\mathrm{y}(t))$ as follows (note $(S)_{ij}$ denotes the ijth element of S; etc.)

$$
\begin{aligned}
(S)_{ii}(\mathrm{y}(t)) &= s_i(\mathrm{y}(t)) && i = 1, \ldots, n; \\
(S)_{ij}(\mathrm{y}(t)) &= 0 && i \neq j,\ i,j = 1, \ldots, n,
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
(P)_{ii}(\mathrm{y}(t)) &= (1 - p_i(\mathrm{y}(t))) && i = 1, \ldots, n; \\
(P)_{i+1i}(\mathrm{y}(t)) &= p_i(\mathrm{y}(t)) && i = 1, \ldots, n - 1; \\
(P)_{ij}(\mathrm{y}(t)) &= 0 && j \neq i, i + 1,\ i,j = 1, \ldots, n.
\end{aligned}
\tag{18}
$$

Define an input vector $\mathbf{f}(\mathrm{y}(t)) = (f_1(\mathrm{y}(t)), \ldots, f_n(\mathrm{y}(t)))'$, where $f_1(\mathrm{y}(t))$ is a scalar function representing the number of trees entering the smallest stage through natural regeneration during time interval t, and $f_i = 0$, $i = 2, 3, \ldots, n$. The element $u_i(t)$ of the control vector $\mathbf{u}(t)$ in expression (2) represents the number of trees harvested from the ith stage at the end of time period t, and the dimension of $\mathbf{u}(t)$ is equal to n. The survival and transition matrices, the regeneration vector, and the state and control vectors are combined into the following difference equations for a stand with an initial state $\mathbf{x}(0) = \mathbf{x}_0$:

$$
\begin{aligned}
\mathbf{x}(1) &= \mathbf{x}(0) - \mathbf{u}(0) \\
\mathbf{x}(t + 1) &= P(\mathrm{y}(t))S(\mathrm{y}(t))\mathbf{x}(t) + \mathbf{f}(\mathrm{y}(t)) - \mathbf{u}(t) \qquad t = 1, 2, \ldots
\end{aligned}
\tag{19}
$$

The structure of this matrix model is an extension of age-structured Leslie matrix models where age classes are replaced by growth stages and where the transition from one stage to the next in each time period is incomplete; that is, subdiagonal elements appear in the transition matrix. Note that the model incorporates measures of stand density and density-dependent functions for predicting regeneration, growth, and survival. Stand growth models that can be represented by Equation (19) include density-dependent matrix models (e.g., Adams and Ek 1974) and fixed parameter matrix models with density-dependent regeneration (e.g., Michie and McCandless 1986). The growth matrix P in expression (18) can be expanded to grow trees more than one stage during the projection interval by adding additional off-diagonal elements (e.g., Michie and McCandless 1986). The growth matrix P can also be expanded to project stands with multiple species, as discussed in a later section (see also Solomon et al. 1986).

Equilibrium Endpoint Constraints

The numerical solution method involves converting the equilibrium endpoint problem (6), which has a terminal steady-state constraint, into an equivalent unconstrained problem with a free terminal point. This is accomplished by adding a penalty to the objective function for any violation of the equilibrium constraint. In the context of problem (6), an optimal sequence of harvests $\tilde{\mathbf{u}}(t)$, $t = 0, 1, \ldots, T - 1$, is sought subject to the growth dynamics (19) and the terminal stand structure $\mathbf{x}_\mathrm{u}$ satisfying the equilibrium condition (4). From equilibrium constraint (4) and growth dynamics (19) it follows that the steady-state harvest vector $\mathbf{u} = \mathbf{h}(\mathbf{x}_\mathbf{u})$ satisfies

$$\mathbf{h}(\mathbf{x}_\mathbf{u}) = P(\mathbf{y})S(\mathbf{y})\mathbf{x}_\mathbf{u} + \mathbf{f}(\mathbf{y}) - \mathbf{x}_\mathbf{u}, \tag{20}$$

and $\mathbf{h}(\mathbf{x}_\mathbf{u}) \geqslant 0$. To ensure that the latter constraint is satisfied, let $\mathbf{g}(\mathbf{x}_\mathbf{u})$ be the penalty vector containing elements g_i, $i = 1, 2, \ldots, n$, that are scalar functions defined by

$$g_i(\mathbf{x}_\mathbf{u}) = \begin{cases} 0 & \text{if } h_i(\mathbf{x}_\mathbf{u}) \geqslant 0 \\ h_i^2(\mathbf{x}_\mathbf{u}) & \text{if } h_i(\mathbf{x}_\mathbf{u}) < 0. \end{cases} \tag{21}$$

Thus, when the steady-state harvest level in stage i is nonnegative, the penalty function g_i is zero, and when the harvest level is negative, the penalty function equals the square of the harvest level. Note that g_i is a continuous, nonnegative function of the terminal state $\mathbf{x}_\mathbf{u}$. The penalty vector is added to the objective function (6) by selecting a suitable vector $\mu \in R_n$ of positive constants; that is, the augmented objective function for the equilibrium endpoint problem becomes

$$\max_{\{\mathbf{u}(t), t=0,1,\ldots,T-1\}} J_T = \sum_{t=0}^{T-1} \delta^t R(\mathbf{x}(t),\mathbf{u}(t)) + \frac{\delta^T}{1-\delta} R(\mathbf{x}_\mathbf{u},\mathbf{h}(\mathbf{x}_\mathbf{u})) - \mu\mathbf{g}(\mathbf{x}_\mathbf{u}). \tag{22}$$

Note that the terminal state $\mathbf{x}_\mathbf{u}$ must also satisfy

$$\mathbf{x}_\mathbf{u} = P(\mathbf{y}(t-1))S(\mathbf{y}(t-1))\,\mathbf{x}(t-1) + \mathbf{f}(\mathbf{y}(t-1)) - \mathbf{u}(t-1). \tag{23}$$

Thus, augmented problem (22) is solved for a given penalty function parameter μ, subject to growth dynamics (19) holding for $t = 0, 1, \ldots, T-1$, and with the terminal state $\mathbf{x}_\mathbf{u}$ free.

Fixed Endpoint Constraints

Fixed endpoint problems are solved by defining a penalty vector that penalizes the objective function whenever the desired terminal state is not reached. In the context of problem (13) an optimal sequence of harvests $\tilde{\mathbf{u}}(t)$, $t = 0, 1, \ldots, T-1$, is sought subject to the growth dynamics (19) and the constraint that the terminal state $\mathbf{x}(T) = \mathbf{x}_\delta$, the extremal steady state. Of course, any steady state can be used as a target in this formulation. To ensure that this constraint is satisfied, let $\mathbf{g}(\mathbf{x}(T))$ be the penalty vector containing elements g_i, $i = 1, 2, \ldots, n$, that are scalar functions defined by

$$g_i = (x_i(T) - x_{i\delta})^2. \tag{24}$$

Note that whenever the terminal state $x_i(T)$ does not equal the desired state $x_{i\delta}$ the penalty function g_i is positive. The penalty vector is added to the objective function (13) by selecting a suitable vector $\mu \in R^n$ of positive constants, that is, the augmented objective function becomes

$$\max_{\{\mathbf{u}(t), t=0,1,\ldots,T-1\}} I_T = \sum_{t=0}^{T-1} \delta^t R(\mathbf{x}(t),\mathbf{u}(t)) + \frac{\delta^\mathrm{T}}{1-\delta} R(\mathbf{x}(T),\mathbf{u}(T)) - \mu\mathbf{g}(\mathbf{x}(T)). \tag{25}$$

Since $\mathbf{x}(T)$ is a function of the state and control in period $T-1$ [see Equation (23)], augmented problem (25) is solved for a given penalty function parameter μ, subject to growth dynamics (19) holding for $t = 0, 1, \ldots, T-1$, and with the terminal state $\mathbf{x}(T)$ free. Since each g_i is minimized when $\mathbf{x}(T) = \mathbf{x}_\delta$ [see Equation (24)] by subtracting $\mu\mathbf{g}(\mathbf{x}(T))$ as in Equation (25), the maximum solution should yield $\mathbf{x}(T) = \mathbf{x}_\delta$ provided $\mathbf{x}_\delta$ is feasible (that is, the system can be driven from $\mathbf{x}_0$ to $\mathbf{x}_\delta$ over $[0,T]$ using controls $\mathbf{u} \in U$).

Sequential Solution Procedure

For a given penalty function parameter μ, the augmented problems (22) and (25) can be solved using gradient techniques (see Haight et al. 1985). A gradient algorithm starts with an initial guess for the control variables $\mathbf{u}(t)$, $t = 0, 1, \ldots, T-1$, and seeks to improve the objective function value by making successive approximations of the optimal control variable values. Each approximation is obtained by moving in the direction of the gradient of the control variables, and the algorithm terminates when improvements in the objective function value are less than a set tolerance. The solution obtained when the algorithm terminates approximates a stationary point that satisfies Kuhn-Tucker optimality conditions.

In practice, it is possible to get arbitrarily close to the optimal solution to the original constrained problem by computing the solution to the augmented problem for sufficiently large μ. However, for very large values for μ, more emphasis is placed on feasibility and the gradient solution method moves rapidly toward a feasible point. Typically a solution is reached that is far from optimal, but movement away from the point is difficult because of the size of the penalty function, and as a result, premature termination of the solution algorithm takes place. This problem is avoided by solving a sequence of problems for increasing penalty function parameters. With each new value for μ, the gradient method is employed starting with the optimal solution corresponding to the previous parameter problem. The solution to each problem is generally infeasible, but as μ is made large, the solutions approach the optimal solution to the original constrained problem.

The sequential solution method is summarized as follows (see Bazaraa and Shetty 1979, p. 341). To initialize the algorithm, choose a termination scalar $\epsilon > 0$, initial guesses for control variables $\mathbf{u}^0(t)$, $t = 0, 1, \ldots, T-1$, initial vector of penalty function parameters $\mu^0 > \mathbf{0}$, and a scalar $\beta > 1$. Set the counter $k = 0$.

1. Solve the augmented problem starting with $\mathbf{u}^k(t)$, $t = 0, 1, \ldots, T-1$ and μ^k. Let the solution be $(\mathbf{x}^{k+1}(t), \mathbf{u}^{k+1}(t))$, $t = 0, 1, \ldots, T-1$ and $\mathbf{x}^{k+1}(T)$.
2. If $\mu^k\mathbf{g}(\mathbf{x}^{k+1}(T)) < \epsilon$, stop. Otherwise, let $\mu^{k+1} = \beta\mu^k$, replace k by $k+1$ and go to step one.

A STAGE-STRUCTURED MODEL FOR TRUE FIR

Stage Definition

To demonstrate the solution method, we have constructed a stage-structured model for projecting growth and yield in mixed white fir and red fir stands. The model is constructed using equations obtained from the California Conifer Timber Output Simulator (CACTOS), a single-tree simulator for mixed conifer stands in Northern California (Wensel and Koehler 1985). CACTOS includes equations that predict five-year changes in the diameter, height, and height to crown base of each tree in a list that describes the stand. Each tree record also includes an expansion factor for the number of trees that the record represents. The growth equations depend on measures of stand crown cover that are obtained by summing the products of tree crown covers and tree factors over all trees in the list. Tree volume is computed by summing the volumes of 16.5 ft logs that can be cut from the tree, and total stand volume is obtained by summing the products of total tree volumes and tree factors.

The stage-structured model is constructed by first defining growth stages for each species. The state vector **x** in growth Equation (19) is divided into two sets $(x_1, \ldots, x_{18})$ and $(x_{19}, \ldots, x_{36})$ representing the numbers of trees in white fir and red fir growth stages, respectively. For management analysis, trees are separated into sapling, pole, and sawtimber stages. The sapling stages, (x_1, x_2, x_3) for white fir and (x_{19}, x_{20}, x_{21}) for red fir, represent the numbers of trees in two-ft height classes that range between 0 and 6 ft. The pole and sawtimber stages, $(x_4, \ldots, x_{18})$ for white fir and $(x_{22}, \ldots, x_{36})$ for red fir, represent the numbers of trees in 2-in. diameter (at breast height) classes. Trees in the ith stage are assigned the midpoint diameter d_i where $(d_4 = 1, d_5 = 3, \ldots, d_{18} = 29)$ for white fir and $(d_{22} = 1, d_{23} = 3, \ldots, d_{36} = 29)$ for red fir. Two-in.-diameter classes were chosen because the potential diameter growth of white fir and red fir trees growing in stands with moderate site quality is less than two inches in five years (Wensel and Koehler 1985). The maximum tree size is limited by the data used to construct the white fir and red fir growth equations in CACTOS (see Biging 1984).[1]

Tree diameter is a convenient attribute for defining growth stages because it can be related to tree height, volume, and crown attributes. The average tree height was computed for each stage using equations described by Van Deusen and Biging (1985). Average tree volume was computed with the assumption that trees in each stage are sectioned into 16.5 ft logs starting at a 1.0 ft stump height. The top diameter and volume of each log were computed with equations given by Biging (1984). Logs that have top diameters less than 6.0 in. were assumed to be unmerchantable. Trees less than 7.0 in. in diameter (at breast height) were assumed to be less than 17.5 ft tall and, thus, did not contain any logs. Average tree volumes were computed by summing log volumes. The average volume of trees in each red fir growth stage differed by less than 10% from the average volume of trees in the corresponding white fir stage.

The models for diameter growth and regeneration depend on measures of the cross-sectional crown area of the stand. The ith element of the stand density vector **y** is defined as the cross-sectional crown area of the stand measured at 66% of the height of trees in stage i. Thus, A in Equation (16) is a 36×36 matrix, and the ijth element of A represents the cross-sectional area of trees in stage j measured at 66% of the height of trees in stage i. The elements of the jth column of A represent a profile of the cross-sectional crown areas of a tree in stage j. These elements were calculated using equations for crown volume, height to crown base, and crown area given by Wensel and Koehler (1985). Compared to the crown profile of a white fir tree in a given diameter class, the crown areas of a red fir tree in the same diameter class are smaller at the base of the crown and larger near the top of the crown.

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[1] In the stage-structured model constructed here, CACTOS diameter growth equations are applied to trees between 1 and 5 in. in diameter. The minimum tree size in data set used to construct the diameter growth equations was 6 in. (Biging 1984). Thus, the projections of pole-size trees should be viewed with caution.

The structure of the growth matrix P in expression (12) is modified to project the two-species state vector $\mathbf{x}$ (see Appendix). The elements of P include upgrowth functions ($p_1, \ldots, p_{18}$) and ($p_{19}, \ldots, p_{36}$) for the white fir and red fir growth stages. Trees are projected to grow between pole and sawtimber stages using diameter growth models given by Wensel and Koehler (1985). Diameter class transition rates are the products:

$$p_i = \psi_i \omega_i(\mathrm{y}) \qquad i = 4, \ldots, 18, 22, \ldots, 36 \tag{26}$$

where ψ_i is the maximum proportion of trees in stage i that move up one diameter class under no competition, and $\omega_i(\mathrm{y})$ is the percentage of the maximum growth percent that can be achieved with a given level of competition. The potential diameter class transition rates are time invariant and depend on the productive capability of the site. In the simulations that follow, the potential growth rates are the same for each species and are computed using a site index of 60 (height in feet at age 50) (see Wensel and Koehler 1985).

The impact of density-dependent competition on potential white fir and red fir diameter class transition rates is:

$$\omega_i(\mathrm{y}) = \begin{cases} e^{-2.544\left(\frac{y_i}{43560}\right)^{0.8458}} & \text{if } i = 4, \ldots, 18 \\ e^{-10.500\left(\frac{y_i}{43560}\right)^{0.1941 c_i - 0.28}} & \text{if } i = 22, \ldots, 36 \end{cases} \tag{27}$$

where y_i is the stand crown cover measured at 66% of the height of trees in stage i, and c_i is the average crown volume for trees in stage i. The competition functions for white fir depend on stand density while the functions for red fir depend on crown volume in addition to stand density. Red fir crown volumes were computed using equations from Wensel and Koehler (1985). All levels of crown cover produce greater reductions in the growth rates of red fir with diameters less than 7 in. compared to the impacts of crown cover on white fir diameter growth (Figure 1). Conversely, crown covers greater than 40% produce smaller reductions in the growth rates of red fir greater than 15 in. in diameter than do the same levels of crown cover on white fir diameter growth. These differences in the shapes of the competition functions have a profound impact on the species composition of optimal uneven-aged management regimes. Note that even small levels of crown cover cause immediate reductions in diameter growth. A reviewer pointed out that intertree competition may take place only after a significant amount of crown cover is present; perhaps as much as 30%. In this case, the competition models would be sigmoid and asymptotic to the upper axis. Because of the apparent uncertainty in the form of the competition model and because of the sensitivity of the optimal solutions to the form of the model, the stand structures and species compositions of optimal management regimes should be viewed with caution. Finally, trees in each species are not allowed to grow beyond the 29-in. growth stage so that $p_{18} = 0.0$ and $p_{36} = 0.0$.

The survival matrix S in expression (17) includes survival functions s_i for the proportion of trees in stage i that survive during a five-year projection interval. Survival equations based on observations of white fir and red fir survival rates were not available. Instead, the survival rates for poles and sawtimber were projected with the following equation,

$$s_i = 1.0 - 0.1e^{-0.109 d_i} \qquad i = 4, \ldots, 18, 22, \ldots, 36, \tag{28}$$

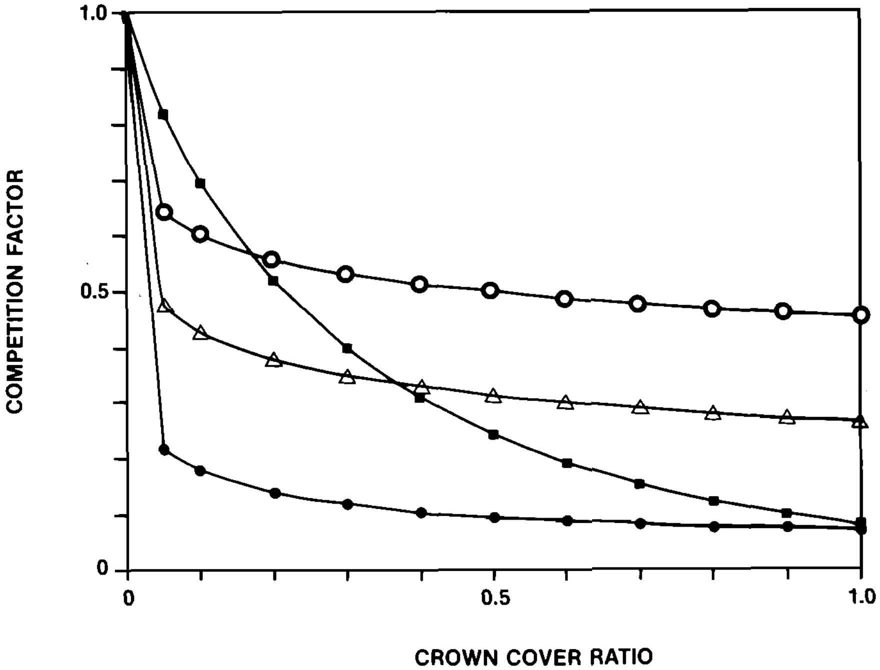

FIGURE 1. The competition factors for white fir (■) and red fir with dbh equal to 29 in. (○), 15 in. (△), and 7 in. (●) are plotted against the crown cover ratio, which is the proportion of the ground area covered by the crowns of trees in the stand. The competition factor is the percentage of the potential diameter growth that can be achieved with a given level of competition as measured by the crown cover ratio [see Equations (26) and (27)].

which was adapted from a survival equation for Douglas-fir in the North Coast region of California given by Wensel and Koehler (1985). In this formulation of the stage-structured model, mortality is not captured in periodic harvests. The model could be formulated to include the projected mortality in each five-year harvest, which would result in positive harvests in all merchantable diameter classes.

SAPLING HEIGHT GROWTH, SURVIVAL AND REGENERATION

Observations of the height growth of saplings growing in openings that were created by cutting overstory trees indicated that white fir and red fir can grow to 6 ft in less than 15 years (Gordon 1973). Thus, in the stage-structured model, we assume that saplings of both species grow into the next stage independently of stand density; that is, $p_i = 1.0$, $i = 1, 2, 3, 19, 20, 21$. In the same study, Gordon (1973) observed that about 30% of the white fir and red fir saplings died during a five-year interval. Thus, in the stage-structured model, we assume that $s_i = 0.7$, $i = 1, 2, 3, 19, 20, 21$. The sapling growth and survival observations were made in openings greater than 8 ac in size. In uneven-aged stands, openings are usually less than 1 ac, and as a result, competition from neighboring trees is likely to affect growth. Thus, the density-independent models defined here may overestimate the sapling growth and survival rates found in uneven-aged stands.

The regeneration vector f in growth Equation (19) contains the functions

$f_1(\mathbf{y})$ and $f_{19}(\mathbf{y})$ representing the numbers of trees added to the smallest white fir and red fir stages, respectively. These functions were constructed based on observations of natural regeneration in mixed white fir and red fir stands in California (Gordon 1970, Gordon 1979). Each regeneration function is the product of the seedling establishment rate and the total number of seeds produced:

$$f_i = \begin{cases} \tau_i(\mathbf{y})\Sigma_{j=1}^{18}\, \upsilon_j x_j & \text{if } i = 1 \\ \tau_i(\mathbf{y})\Sigma_{j=19}^{36}\, \upsilon_j x_j & \text{if } i = 19 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

where τ_1 and τ_{19} are the seedling establishment rates for white fir and red fir, respectively, and υ_j is the number of seeds produced by a tree in stage j. The seedling establishment rates for white fir and red fir depend on y_1, the cross-sectional crown area of the stand measured at ground level:

$$\tau_i = \begin{cases} \dfrac{0.0711y_1}{43560}\left(1.0 - \dfrac{y_1}{43560}\right)^3 & \text{if } i = 1 \\[2ex] \dfrac{0.1220y_1}{43560}\left(1.0 - \dfrac{y_1}{43560}\right)^5 & \text{if } i = 19. \end{cases} \tag{30}$$

The white fir establishment rate peaks at 25% crown cover, and the red fir establishment rate peaks at 18% crown cover, thereby modeling the observation that red fir is slightly less tolerant of shade than white fir. Seed production per tree depends on tree size, and trees less than 15 in. in diameter do not produce seed:

$$\upsilon_j = \begin{cases} 41.32(d_j - 13.0)^2 & \text{if } j = 11, \ldots, 18, 29, \ldots, 36 \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

The transition and survival matrices P and S, and the regeneration vector $\mathbf{f}$ are embedded in the growth dynamics equation (19) to update stand structures during five-year projection intervals.

Discussion

While the stage-structured model presented above is constructed for the purpose of developing and comparing solutions to the fixed and equilibrium endpoint management problems, it also demonstrates a method for simplifying a single-tree simulator. Single-tree simulators such as CACTOS are not suited for iterative optimization algorithms because projecting the attributes of a large number of tree records requires too much execution time and computer storage. In contrast to a single-tree simulator, a stage-structured model contains fewer state variables and growth equations, and exact expressions for model derivatives can be written and embedded in gradient-based optimization algorithms. As a result, optimal solutions to uneven-aged management problems can be obtained with less execution time.

While a stage-structured model may be more tractable for optimization, there are differences in its structure relative to a single-tree simulator that may cause differences in stand projections. In the stage-structured model constructed here, trees are classified into diameter and species classes, and each class represents a growth stage that is further described by average tree height, volume, and crown attributes. These class attributes are fixed for the course of the projection. Trees are assumed to be uniformly distributed across each diameter class, and a proportion of the trees in each stage are projected to grow into the next stage using a density-dependent diameter growth model. In contrast to the stage-structured model, CACTOS includes density-dependent models for projecting the diameter, height, and crown dimensions of each tree record. These additional models and the fact that they can be applied to a very large number of tree records should allow CACTOS to make more accurate projections of stand diameter distributions and volumes over time.

The gain in accuracy with CACTOS compared to the stage-structured model is not known. We did compare the volume projections from the two simulators for a wide range of management prescriptions that were applied to hypothetical white fir and red fir stands of moderate site quality (see Haight and Getz, in press). The differences in projections of total volume yields (as a percent of the CACTOS projections) ranged between 0% and 10%. The stage-structured model also projected the same trends in volume yields by product class. These results suggest that the accuracy of the stage-structured model projections of stand volume and product yields will be similar to the accuracy of CACTOS projections. Firm conclusions cannot be reached, however, without comparing simulator projections to real stand growth and yield observations.

OPTIMAL MANAGEMENT REGIMES

HARVEST VALUATION

In the context of the stage-structured model given in Equation (19), the control vector $\mathbf{u}(t)$ contains elements $u_i(t)$, $i = 1, \ldots, 36$, representing the numbers of trees harvested from the ith stage at the end of time period t. The solution method determines the best number of trees to harvest from each merchantable and unmerchantable white fir and red fir growth stage in each time period. The revenue function R in expression (3) is computed by assigning prices to trees harvested in each stage:

$$R(\mathbf{x}(t),\mathbf{u}(t)) = \sum_{i=1}^{36} b_i u_i(t) \tag{32}$$

where b_i is the price per tree in stage i. Saplings and poles are assumed to be unmerchantable and cost \$0.25 per tree to cut. The California State Board of Equalization, which reports average stumpage prices for use in yield tax calculations, priced true fir at \$25 per mbf for the first half of 1986. White fir and red fir sawtimber was assigned this price, and the prices were assumed to be constant over the planning horizons. Values for b_i are the product of stumpage price and tree volume (Table 1). The real discount rate is 4.0%.

EQUILIBRIUM ENDPOINT PROBLEMS

To demonstrate the penalty function methods, we solved the equilibrium endpoint formulation that maximizes criterion (6) and determined the costs of satisfying equilibrium harvest constraints that were imposed after transition periods that varied between 0 and 60 years in length. The problems were solved for the mixed white fir and red fir stand given in Table 1, and the cutting cycle was five years.

The present values of optimal transition and steady-state regimes improve with increasing transition period length and level off at \$440 per ac, the present value of the 60-year transition regime (Figure 2). The present values

TABLE 1. The tree prices and initial diameter distributions listed here were used in the fixed and equilibrium endpoint problems.

Diameter class midpoint (in.)	Price		Diameter distribution	
	White fir	Red fir	White fir	Red fir
	($/tree)		(trees/ac)	
0[a]	−0.25	−0.25	180.0	149.0
1	−0.25	−0.25	115.0	145.4
3	−0.25	−0.25	51.3	47.1
5	−0.25	−0.25	32.2	23.2
7	0.09	0.09	23.2	13.7
9	0.32	0.29	17.0	6.4
11	0.75	0.70	13.7	4.0
13	1.46	1.39	11.3	2.9
15	2.51	2.44	9.7	2.4
17	3.97	3.95	8.8	2.0
19	5.91	5.99	8.1	1.9
21	8.40	8.66	5.0	1.8
23	11.50	12.05	.0	1.7
25	15.28	16.25	.0	1.7
27	19.81	21.36	.0	1.8
29	25.15	27.48	.0	.8
Value[b] ($/ac)			88.8	68.0

[a] The 0-in. diameter class includes trees in three sapling stages: 0 to 2, 2 to 4, and 4 to 6 ft in height, respectively.
[b] Total undiscounted value of growing stock.

of the 0- and 10-year transition regimes are $50 per ac (11%) and $46 per ac (10%) less than the present value of the 60-year regime. The costs of achieving steady states in years 20 to 50 are less than $25 per ac, which is 6% of the present value of the 60-year regime.

The steady-states associated with the 0- and 60-year transition regimes have different species compositions (Table 2). The steady state obtained in year 60 includes only white fir. During transition harvesting, red fir is cut when it becomes merchantable, and the remaining unmerchantable red fir is cut in year 60, prior to the establishment of the steady state. The steady state obtained in year 0 includes both species. In this case, red fir is not completely liquidated because of the cost of cutting the unmerchantable trees, and merchantable red fir trees are kept in the stand to satisfy the steady-state requirement. The steady state obtained in year 10 includes white fir in higher proportions, and the steady-states obtained in years 20 to 50 include only white fir. As the transition period increases, the value of the steady-state yield decreases from $46.90 per ac ($T = 0$) to $45.40 per ac ($T = 60$).

The equilibrium endpoint problems were relatively difficult to solve because the penalty vector included functions for the stage-class growth dynamics [note Equations (20) to (22)]. The penalty function algorithm involved solving a sequence of unconstrained optimization problems with increasing penalty function parameters. We terminated the algorithm when either $\mu g(x_u) < .01$ or $\mu > 10{,}000$. In the former case, a feasible solution had been reached, while in the latter case, the algorithm had converged to an

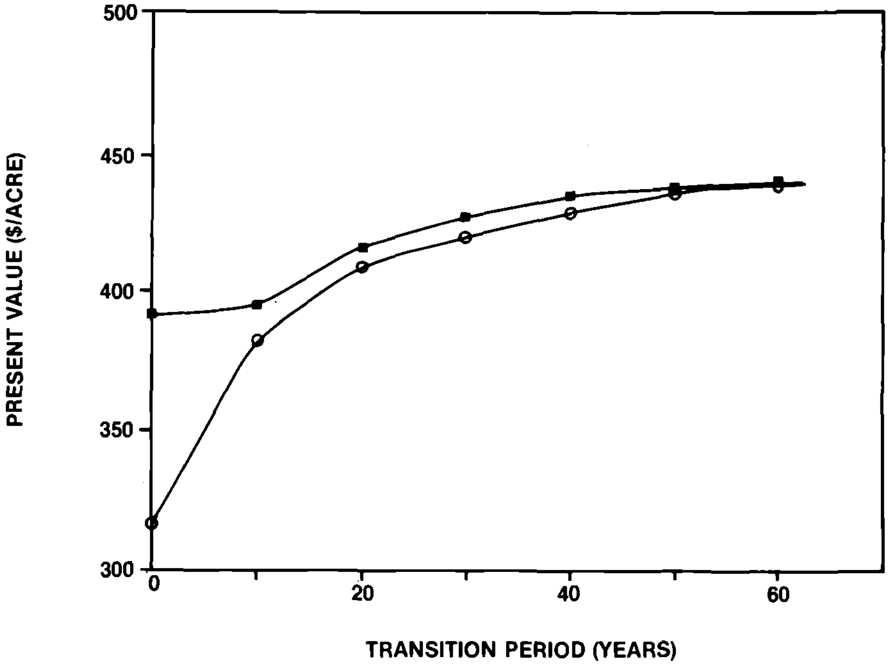

FIGURE 2. The present values of transition and steady state management regimes that solve the equilibrium endpoint formulation in expression (6) (■) and the fixed endpoint formulation in expression (13) (○) are plotted against transition period length. The fixed endpoint is the extremal steady state given in Table 3, and the initial stand is the mixed white fir and red fir stand described in Table 1. The cutting cycle is five years.

infeasible solution. The solutions obtained at termination were most sensitive to the initial guesses given to control variables, $\mathbf{u}^0(t)$, $t = 0, \ldots, T - 1$ and the initial size of μ, the penalty function parameter. Random initial guesses for the control variables caused infeasible termination; however, many of the solutions were nearly feasible. Using a near-feasible solution as the initial guess for the control variables usually resulted in a feasible termination. For any near-feasible initial guess, feasible solutions were obtained with $\mu^0 = 0.1$. Starting with penalty parameters substantially less than 0.1 resulted in infeasible termination because the penalty function did not affect the determination of the control variable values. Starting with penalty parameters substantially greater than 0.1 resulted in suboptimal solutions because the penalty function dominated the selection of the optimal control variable values. Several near-feasible or feasible initial guesses for the control variables were explored before we were confident that an optimal solution had been obtained.

Fixed Endpoint Problems

The fixed endpoint problem given by expression (13) involves the determination of the optimal harvest policy for a finite transition period that terminates in a predefined steady-state policy. Three kinds of target steady states were computed (Table 3): the extremal steady state (ESS), the investment-efficient steady state (IESS), and the maximum sustainable rent (MSR) steady state. Which criterion is chosen has a big effect on the present value and species composition of transition and steady-state management.

TABLE 2. The optimal steady-state diameter distributions listed here were obtained with equilibrium endpoint constraints imposed after 0 and 60 years of transition harvesting in a mixed white fir and red fir stand. The cutting cycle is 5 years.

Diameter class midpoint (in.)	Year 0				Year 60			
	White fir		Red fir		White fir		Red fir	
	Stock[a]	Yield	Stock	Yield	Stock	Yield	Stock	Yield
	trees/ac							
0[b]	179.0	0.0	147.0	0.0	185.0	0.0	0.0	0.0
1	95.7	.0	137.1	.0	90.0	.0	.0	.0
3	46.2	.0	46.8	.0	42.7	.0	.0	.0
5	30.6	.0	23.2	.0	27.6	.0	.0	.0
7	20.3	2.1	12.8	.0	20.9	.0	.0	.0
9	13.6	.0	6.1	.0	16.0	.0	.0	.0
11	10.6	.0	3.8	.0	13.2	.0	.0	.0
13	7.8	.0	2.7	.0	11.1	.0	.0	.0
15	7.0	.0	2.2	.0	9.8	.0	.0	.0
17	6.4	.0	1.8	.0	9.0	.0	.0	.0
19	5.7	1.1	1.6	.0	7.4	3.7	.0	.0
21	3.3	2.9	1.4	.0	2.8	2.8	.0	.0
23	.3	.3	1.1	.0	.0	.0	.0	.0
25	.0	.0	1.0	.2	.0	.0	.0	.0
27	.0	.0	.5	.3	.0	.0	.0	.0
29	.0	.0	.1	.1	.0	.0	.0	.0
Value[c] ($/ac)	45.5	34.5	11.1	12.4	74.4	45.4	0.0	0.0

[a] Stock denotes the before-harvest stand structure.
[b] The 0-in. diameter class includes trees in three sapling stages: 0 to 2, 2 to 4, and 4 to 6 ft in height, respectively.
[c] Total undiscounted values of before-harvest growing stock and steady-state yield.

The extremal steady state represents the harvesting regime that would be attained after a sufficient number of unconstrained harvests in any stand assuming that biological and economic parameters are constant over time. The ESS for the bionomic model discussed above was found by solving the equilibrium endpoint problem for the two-species stand in Table 1 with transition period lengths that did not affect the resulting steady-state stand structure. The ESS includes white fir trees up to 21 in. in diameter, and harvests take place in the 19- and 21-in. diameter classes (Table 3). The value of the steady-state yield is $40.60 per ac.

The investment-efficient steady state solves the static optimization problem (14). This problem was solved using the penalty function method described above on the following augmented objective function that includes the steady-state harvest level $\mathbf{h}(\mathbf{x}_u)$ and the penalty vector $\mathbf{g}(\mathbf{x}_u)$ [see Equations (20) and (21)]:

$$\max_{\{\mathbf{x}_u\}} \left[\frac{\delta}{1-\delta} R(\mathbf{x}_u,\mathbf{h}(\mathbf{x}_u)) - R(\mathbf{x}_u,\mathbf{x}_u) - \mu\mathbf{g}(\mathbf{x}_u) \right] . \tag{33}$$

The IESS includes red fir trees up to 29 in. in diameter, and harvests take place in the 7- and 29-in. diameter classes (Table 3). The value of the steady-state yield is $50.10 per ac.

TABLE 3. The steady states listed here were used as fixed endpoints to achieve after a finite number of transition harvests in a mixed white fir and red fir stand. The extremal steady state (ESS) contains only white fir while the investment-efficient steady state (IESS) and the maximum sustainable rent steady state (MSR) contain only red fir. The cutting cycle is 5 years.

Diameter class midpoint (in.)	ESS		IESS		MSR	
	White fir		Red fir		Red fir	
	Stock[a]	Yield	Stock	Yield	Stock	Yield
	trees/ac					
0[b]	141.0	0.0	1032.0	0.0	714.0	0.0
1	66.3	.0	936.6	.0	695.0	.0
3	34.5	.0	320.4	.0	222.5	.0
5	23.8	.0	172.1	.0	110.6	.0
7	18.4	.0	38.0	15.2	66.3	.0
9	14.4	.0	11.9	.0	30.9	.0
11	12.0	.0	8.1	.0	19.7	.0
13	10.2	.0	6.2	.0	14.5	.0
15	9.1	.0	5.2	.0	11.8	.0
17	8.5	.0	4.6	.0	10.2	.0
19	6.8	5.3	4.2	.0	9.2	.0
21	1.1	1.1	4.0	.0	8.7	.0
23	.0	.0	3.9	.0	8.4	.0
25	.0	.0	4.0	.0	8.4	.0
27	.0	.0	4.1	.0	8.6	.0
29	.0	.0	1.8	1.8	3.4	3.4
Value[c] ($/ac)	69.8	40.6	−254.4	50.1	327.7	93.4

[a] Stock denotes the before-harvest stand structure.
[b] The 0-in. diameter class includes trees in three sapling stages: 0 to 2, 2 to 4, and 4 to 6 ft in height, respectively.
[c] Total undiscounted values of before-harvest growing stock and steady-state yield.

The maximum sustainable rent steady state maximizes the value of steady-state harvesting and is independent of the discount rate [see criterion (15)]. The MSR steady state was found by solving the following augmented objective function that includes the steady-state harvest level $\mathbf{h}(\mathbf{x}_u)$ and the penalty vector $\mathbf{g}(\mathbf{x}_u)$ [see Equations (20) and (21)]:

$$\max_{\{\mathbf{x}_u\}} \left[\frac{\delta}{1-\delta} R(\mathbf{x}_u, \mathbf{h}(\mathbf{x}_u)) - \mu \mathbf{g}(\mathbf{x}_u) \right]. \tag{34}$$

The MSR steady state includes red fir trees up to 29 in., and all trees growing into the 29-in. diameter class are cut (Table 3). The value of the steady-state yield is $93.40 per ac.

The fixed endpoint problem [expression (13)] was solved for the mixed species stand in Table 1 using the ESS as the target stand structure after transition periods that varied between 0 and 60 years in length. Due to the more restrictive endpoint constraints, the present value of the transition and steady-state management regime is less than the present value of the equilibrium endpoint solution for each transition period (Figure 2). Converting to the ESS in year 0 results in a $75 per ac (19%) reduction in present value relative to the equilibrium endpoint solution because of the costs associated with harvesting unmerchantable red fir trees in the first cut. By year 60 the management regime that converts to the ESS nearly matches the equilibrium endpoint regime.

Feasible solutions to the fixed endpoint problems for the mixed species stand in Table 1 using the IESS and the MSR steady state as targets required more than 200 years of transition harvesting. In each case, merchantable white fir was liquidated and red fir was harvested from the largest diameter class during the transition period. In each case, the present value of the fixed endpoint regime was \$264 per ac (60%) less than the present value of converting to the extremal steady state in 60 years.

Solutions to the fixed endpoint problems were obtained with much less work than were solutions to the equilibrium endpoint problems. The fixed endpoint problems were solved with a penalty function parameter $\mu^0 = 0.1$. Feasible solutions were obtained with random starting values given to the control variables, and optimal solutions did not depend on the initial control variable values.

Discussion

The differences in the diameter growth model parameters for white fir and red fir [see equation (27)] interact with the criteria for steady-state harvesting to cause profound differences in optimal species composition and stand structure. White fir dominates the ESS because any level of stand density competition produces smaller reductions in the growth rates of white fir between 1 and 7 in. in diameter than do the same levels of competition on red fir pole growth rates (see Figure 1). As a result, when maximum present value is the investment criterion, optimal transition harvesting involves liquidating merchantable red fir and building up the white fir growing stock to the extremal steady state.

The investment-efficient criterion seeks the steady state that maximizes the difference between the present value of an infinite series of harvests and the value of the steady-state growing stock. Since trees less than 7 in. in diameter have negative value, steady states with more unmerchantable trees have higher objective function values. Two differences in red fir and white fir diameter growth equations cause the IESS to be dominated by red fir. First, red fir poles less than 7 in. in diameter grow at slower rates than do white fir poles at any level of competition (see Figure 1). As a result, the IESS includes over 2400 unmerchantable saplings and poles that increase the value of the investment-efficient objective function. Second, red fir sawtimber greater than 15 in. in diameter grows faster than does white fir sawtimber for crown covers greater than 40% (see Figure 1). Since red fir crown areas are smaller than white fir crown areas, more large, high valued trees can be produced by growing red fir at high densities compared to growing white fir at the same densities.

The maximum sustainable rent criterion seeks the steady state that provides the highest value harvest and, in contrast to the investment-efficient criterion, does not assess interest costs on the steady-state growing stock. Red fir dominates the MSR steady state because of its superior growth rates in sawtimber diameter classes, and because red fir have smaller crown areas than white fir. As a result, more large, high valued trees can be produced by growing red fir.

Changing the discount rate affected the structure and species composition of the extremal steady state. Discount rates between 1 and 2% resulted in steady states that included both white fir and red fir, and as the discount rate approached zero, the extremal steady state approached the maximum sustainable rent steady state (see Table 3) which includes only red fir. Discount rates greater than 6% resulted in optimal transition regimes in which all trees were harvested before they reached seed-producing size. Optimal harvesting eventually exhausted the growing stock, and as a result, supplemental planting would be required to maintain the stand. Exhausting the growing stock at higher discount rates was optimal in these cases because of the relatively slow growth rates of white fir and red fir on sites with moderate growth potential.

Precommercial thinning was not optimal in any of the solutions presented above, and this result was unaffected by changing the relative costs of cutting submerchantable trees of different sizes. It is likely that density-dependent sapling growth rates would cause precommercial thinning to enter the optimal policy.

CONCLUDING REMARKS

In summary, when the objectives of timber harvesting include the maximization of present value and the achievement of a steady-state harvest policy, the management problem can be formulated as a dynamic harvesting model with fixed or equilibrium endpoint constraints. This paper has focused on issues associated with these constrained problems.

For a given transition period length, the solution to the equilibrium endpoint problem has a higher present value than the solution to any fixed endpoint problem, since the equilibrium endpoint formulation places fewer constraints on the terminal steady state. The equilibrium endpoint policy depends on the initial stand structure and transition period length, and it may differ in terms of species composition and sustainable harvest value from ESS, IESS, and MSR structures. As the transition period lengthens, the equilibrium endpoint policy approaches the ESS, and the cost of the terminal steady-state constraint approaches zero. Numerical solutions showed that the cost of the terminal steady-state constraint can be large (greater than 10% of the present value of the unconstrained solution) for short transition periods, but the impact decreases rapidly as the transition period lengthens.

The cost of a fixed steady-state constraint depends on the criterion used to determine the target steady state and the transition period length. With an ESS target, the solution to the fixed endpoint problem approaches the solution to the equilibrium endpoint problem as the transition period lengthens, and the cost of the steady-state constraint approaches zero. Numerical results showed that the cost of the ESS target in short transition periods can be large (greater than 25% of the present value of the unconstrained solution). The costs of achieving the IESS or the MSR policies can be severe (greater than 60% of the present value of the unconstrained solution) regardless of the transition period length. As the discount rate approaches zero, the ESS policy approaches the MSR policy, and the cost of achieving the MSR policy approaches zero.

The value of the steady-state yield depends on the criterion used to determine the steady-state target. The MSR policy always provides the highest value yield, and those interested in maximizing the productivity of the steady state would want to convert to the MSR policy. In previous studies, the objective associated with the achievement of an IESS structure is the maximization of the present value of harvested yields during the transition to a steady state. It turns out that the IESS structure has the highest investment value (i.e., present value of steady-state yields net investment cost), but this criterion is not consistent with the objective of maximizing present value. In fact, this objective can be achieved more efficiently by solving the equilibrium endpoint problem. The ESS policy may produce a relatively low sustainable yield, but this is offset by the low cost associated with achieving the ESS. The ESS has the property that, once achieved, there exists no transition policy away from the ESS that improves the present value of harvesting.

Achieving steady-state harvesting has always been a desirable goal in uneven-aged management; however, as shown above, there may exist large financial costs associated with steady state constraints. These costs occur because optimal unconstrained harvesting may approach a steady state asymptotically or not at all. More efficient solutions could be obtained by constructing an unconstrained problem that includes the costs of yield fluctuations. For example, the uneven-aged management problem could include capital stock as a state variable with increasing marginal capital adjustment costs. In this case, optimal management would rapidly approach a steady state without explicit constraints.

The methods used here to solve the fixed and equilibrium endpoint problems can be applied to any stage-structured model that includes density-dependent growth functions (e.g., Adams and Ek 1974) or fixed growth parameters (e.g., Michie and McCandless 1986). Whether the optimization methods can be successfully applied to uneven-aged stands that are projected with single-tree simulators is an open question. Results from even-aged stand optimization (Roise 1986) suggest that derivative-free nonlinear programming methods may solve these problems. Nevertheless, as demonstrated here, it is possible to construct a stage-structured model using the relations that are contained in a single-tree simulator. Results from optimizing the stage-structured model can then be used to guide the development of prescriptions with the single-tree simulator.

Much of the modeling work in uneven-aged management has relied on average acre descriptions of stand structure. Diameter distribution descriptions ignore the spatial distribution of trees, and in clumpy stands, they provide very little guidance on the stocking levels to maintan within even-aged tree groups. A model that characterizes an uneven-aged stand by an area versus age distribution and by the average stocking level in groups of each age class may be more realistic.

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APPENDIX

The purpose of the appendix is to describe the growth matrix used in the stage-structured model for white fir and red fir. Recall that the first 18 elements of the state vector $\mathbf{x}$ represent the numbers of trees in white fir growth stages, and the second 18 elements are the numbers of red fir trees. Associated with the state vector is the growth matrix P, which is used in the growth dynamics Equation (19) to compute the movement of trees between size classes:

$$\begin{bmatrix} 1-p_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ p_1 & 1-p_2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & 1-p_3 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_{17} & 1-p_{18} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1-p_{19} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & p_{19} & 1-p_{20} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & p_{20} & 1-p_{21} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & p_{35} & 1-p_{36} \end{bmatrix},$$

where p_i represents the proportion of the trees in stage i that grow into stage $i + 1$ during the five-year projection period.