Formulating a Stand-growth Model for Mathematical Programming Problems in Appalachian Forests

Gary W. Miller and Jay Sullivan[1]

Abstract: Some growth and yield simulators applicable to central hardwood forests can be formulated for use in mathematical programming models that are designed to optimize multi-stand, multi-resource management problems. Once in the required format, growth equations serve as model constraints, defining the dynamics of stand development brought about by harvesting decisions. In large models representing forests with varying cover types equations from several unique simulators can be combined to account for individual stand growth. The inclusion of appropriate growth functions greatly enhances the validity and scope of management guidelines generated by optimization routines. Although several growth models are available for Appalachian hardwoods, few have been formulated to serve as inputs to such analytical models. Methods for formulating nonlinear growth constraints for a two-stage matrix simulator used in certain Appalachian hardwood stands are demonstrated. A generalized growth constraint is presented that may be indexed by size class, species group, and stand for sizable management problems.

INTRODUCTION

Forest growth and yield simulators play an important role in mathematical programming models designed to optimize forest management. In general, optimization models numerically search for a best management strategy from among all feasible alternatives. In forest management models, feasibility is defined by two types of mathematical constraints: landowner management objectives and stand growth capabilities. Equations that represent management objectives simply place limits on harvest decisions so that the optimization procedure considers only the cutting strategies that achieve desired esthetics, wildlife habitat, and so on. Conversely, equations derived from an appropriate stand growth model represent basic biological limits associated with the stand. This type of constraint places upper and lower bounds on residual tree growth resulting from harvest decisions.

Adams and Ek (1974) constructed a nonlinear programming (NLP) problem to solve for optimal structure, stocking, and transition harvesting for uneven-aged northern hardwoods. Within their model, tree growth and mortality over time were expressed as nonlinear functions of initial tree size and total stand basal area at the beginning of each growth period. The model searched for the optimal number of trees in each d.b.h. class at the beginning of a growth period such that the value of periodic growth was maximized. Martin (1982) developed investment-efficient stocking guides for uneven-aged northern hardwood stands using the same growth model and a Weibull distribution function to reduce the number of decision variables.

[1]U.S. Forest Service, Northeastern Forest Experiment Station, Timber and Watershed Laboratory, P.O. Box 404, Parsons, WV 26287; and Virginia Polytechnic Institute, and State University, Department of Forestry, Blacksburg, VA 24061.

Buongiorno and Michie (1980) later used the same plot data (Adams and Ek 1974) to estimate survival and growth probabilities for each diameter class, and to express growth capabilities in a fixed coefficient matrix. The matrix simply transformed an initial tree list to a new list 5 years later, thereby accounting for tree growth and mortality. By employing a matrix of transition probabilities, instead of nonlinear functions, the management model could be constructed as a linear programming (LP) problem. Although LP formulations are usually easier to solve, the fixed matrix does not change in response to cutting decisions, thus limiting its usefulness to problems involving narrow fluctuations in residual stocking.

To date, several simulators are available for eastern hardwood timber types: SILVAH for Allegheny hardwoods (Marquis and Ernst 1992), OAKSIM for even-aged upland oaks (Hilt 1985), NE-TWIGS for mixed eastern hardwoods (Teck 1990), and FIBER for spruce-fir and northern hardwoods (Solomon, Hosmer, and Hayslett 1987). Most available growth models were not designed to function as components of optimization models. Instead, they were developed to allow a user to input stand data, simulate stand development, and project volume and value growth in response to a specific cutting strategy. As a result, the structure of equations within the growth model that represent ingrowth, survival, diameter growth, and mortality often do not conform to requirements of algorithms used to solve mathematical programming problems.

This report demonstrates how equations from FIBER (Solomon, Hosmer, and Hayslett 1987) were used to construct growth constraints within a nonlinear programming problem to optimize individual stand management. FIBER was chosen because its relatively simple structure is amenable for use in nonlinear programming problems. Methods discussed are intended to help users construct similar models for forest management problems where FIBER is an accurate growth projection system.

GENERAL MANAGEMENT MODEL

A general model is described that can be used to optimize harvest schedules for a range of resource objectives. Forest managers need detailed cutting prescriptions that account for the interaction of size, species, and other product-value factors when planning harvests in individual stands. Cutting guides broadly defined by target residual basal area or volume per acre do not adequately describe how key economic factors such as size and species should be controlled to optimize management. Therefore, a basic requirement for this type of timber-oriented optimization model is that resulting prescriptions must define the number of trees to remove in each size class and species group over time.

Haight (1987) presented a model for uneven-aged management that solved for optimal cut and residual diameter distributions over a sequence of periodic harvests in a single stand. A more general version of the model, presented later by Getz and Haight (1989), is a discrete-time optimal control formulation that defines harvest sequences in terms of the number of trees to cut and leave in each size class. The state variable $x_{i,j,k}(t)$ defines the number of trees per acre at the beginning of time t, in size class i, in species group j, in stand k. The control variable $h_{i,j,k}(t)$ represents the number of trees per acre to harvest at time t, in size class i, in species group j, in stand k. The control vector $\mathbf{h}(t)$ could be broken down into roundwood products such as pulpwood, sawtimber, or veneer. For this discussion, however, $\mathbf{h}(t)$ represents only the harvest of merchantable sawtimber, commercial species 11.0 inches d.b.h. and larger.

The Objective Function

Optimization models include an objective function that simply measures the desirability of a particular solution. The solution algorithm systematically adjusts the level of the decision variables until no other solution has a higher objective function value. In this case, the objective function maximizes the present value of all harvests starting with an initial forest $\mathbf{x}(0)=\mathbf{x}_0$ as defined in equation 1.

$$Max \quad NPV = \sum_{i=1}^{MD}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{t=1}^{T} \delta^t \, p_{i,j,k}(t)\, h_{i,j,k}(t) - \sum_{t=1}^{T}\sum_{k=1}^{K} \delta^t \, fc_k(t) \tag{1}$$

The first term sums the product of price $p_{i,j,k}$ times number of trees harvested $h_{i,j,k}$, and discounts each harvest revenue to time $t=0$, where δ is equal to the discount factor $1/(1+r)$ and r is a positive annual discount rate. MD, J, K, and T represent the maximum diameter class, number of species groups, number of stands, and number of time periods, respectively. The second term is a fixed cost $fc_k(t)$ associated with each harvest discounted to $t=0$. This general formulation can be expanded to include variable harvest cost, value of ending inventory, planting costs, and so on.

Stand Growth Constraints

Stand growth constraints take the general form of equations 2 and 3. The number of trees at the beginning of the planning period $\mathbf{x}(0)$ is given as $\mathbf{x}_0$ in equation 2. In subsequent time periods, the initial number of trees $\mathbf{x}(t+1)$ is defined by two components given in equation 3. The first term $G(\mathbf{x}(t), \mathbf{h}(t))$ is a function that estimates growth of the residual stand from the previous period. The second term $F(\mathbf{x}(t), \mathbf{h}(t))$ is a function that estimates ingrowth into the smallest d.b.h. class from the previous period.

$$\mathrm{x}(0) = \mathrm{x}_0 \tag{2}$$

$$\mathrm{x}(t+1) = G(\mathrm{x}(t),\mathrm{h}(t)) + F(\mathrm{x}(t),\mathrm{h}(t)) \qquad t = 0,1,2,\ldots,T \tag{3}$$

Growth and yield simulators applicable to central hardwood forests can be formulated for use in a model of this type designed to optimize stand management. Transformed equations from the simulator take the form of equations 2 and 3 and serve as model constraints which define the dynamics of stand development brought about by harvesting decisions. In large models representing forests with varying cover types equations from several distinct simulators can be combined to account for individual stand growth.

Feasibility and Nonnegativity Constraints

Nonnegativity and feasibility constraints are defined by equations 4, 5, and 6.

$$\begin{aligned} h(t) &\geq 0 \\ x(t) &\geq 0 \end{aligned} \qquad t=0,1,2,\ldots,T \tag{4}$$

$$x_0 - h(0) \geq 0 \qquad t=0 \tag{5}$$

$$x(t) - h(t) \geq 0 \qquad t=1,2,\ldots,T \tag{6}$$

Equation 4 assures that all initial stands and harvests are nonnegative, and equation 5 assures that the initial harvest does not exceed stand stocking at time $t=0$, the initial stand state. Equation 6 assures that harvests do not exceed the initial stand in any time period t. Management constraints to achieve specific objectives can be added to this group of equations to further define feasibility.

FORMULATING GROWTH CONSTRAINTS

In this study, stand growth constraints corresponding to equation 3 were derived from FIBER, a two-stage matrix stand growth simulator originally developed for spruce-fir and northern hardwood types in the Northeast (Solomon, Hosmer, and Hayslett 1987). Growth functions in FIBER were developed from empirical studies to provide a mathematical framework for estimating growth dynamics that may result from management decisions. In general, stand growth is projected with a stage structured model that transforms the residual stand vector $\mathbf{x}(t)$-$\mathbf{h}(t)$ to a new initial stand vector $\mathbf{x}(t+1)$ in the future using a transition matrix whose elements are nonlinear functions of $\mathbf{x}(t)$ and $\mathbf{h}(t)$. These nonlinear growth functions estimate the probability that trees in the initial stand die, survive, and grow into a larger d.b.h. class, or survive without growing into a larger size class.

Basics of FIBER

Regression equations in FIBER employ initial basal area $IBA_k(t)$, residual basal area $RBA_k(t)$, diameter class midpoint D_i, proportion of total basal area comprising hardwood species in stand k, $PH_k(t)$, and proportion of total basal area comprising species j in stand k, $PS_{j,k}(t)$ at time t to predict transition and mortality probabilities over a 5-year period. For example, $u_{i,j,k}(t)$, the probability of a tree in d.b.h. class i, species j, stand k growing into d.b.h. class $i+1$ during a growth period beginning at time t is given by equation 7, where $\beta_{u,j}$ are estimated upgrowth coefficients for each species group j.

$$u_{i,j,k}(t) = \beta_{u,j}\ (IBA_k(t),\ RBA_k(t),\ PH_k(t),\ D_i) \tag{7}$$

Similarly, a transition probability is computed for $a_{i,j,k}(t)$, survival but no upgrowth in equation 8.

$$a_{i,j,k}(t) = \beta_{a,j}\ (IBA_k(t),\ RBA_k(t),\ PH_k(t),\ D_i) \tag{8}$$

$$f_{6j,k}(t) = \beta_{fj}\ (RBA_k(t),\ PH_k(t),\ PS_{j,k}(t)) \tag{9}$$

Mortality for the growth period is estimated implicitly as $1\text{-}a_{i,j,k}(t)\text{-}u_{i,j,k}(t)$. Ingrowth into the smallest size class, in this case 6 inches d.b.h., is given by equation 9. Transition probabilities are then placed into a matrix $\mathbf{G}_{j,k}(t)$ that is applied to a residual stand vector $\mathbf{x}_{j,k}(t)\text{-}\mathrm{h}_{j,k}(t)$ plus ingrowth defined by $f_{6j,k}$ to generate a stand vector $\mathbf{x}_{j,k}(t+5)$ by equation 10. Note that stand growth is dependent on harvest decisions as they affect species composition and residual basal area in each time period.

$$\mathrm{x}_{j,k}(t+5) = \mathrm{G}_{j,k}\ (\mathrm{x}_{j,k}(t) - \mathrm{h}_{j,k}(t)) + f_{6j,k} \tag{10}$$

In summary, FIBER performs the growth simulation in two stages. In the first stage, elements of $\mathbf{G}(t)$ and $\mathbf{F}(t)$ are computed based on number of trees in each size class and species group before and after a harvest in each stand. In the second stage, a linear operation is performed according to equation 10 to estimate a new initial stand structure $\mathbf{x}(t+5)$ 5 years in the future. At each 5-year interval, the two-stage procedure is repeated so that growth is always a function of stand density and species composition at the beginning of each time period.

FIBER in Nonlinear Programming

For nonlinear programming problems, growth functions from FIBER are formulated as constraints that place upper and lower bounds on productivity of the stand, one growth constraint for each combination of diameter class i, species group j, and stand k. For example, the general constraint for $x_{18j,k}(t+5)$, the number of trees in the 18-inch d.b.h. class in species j, stand k, at time $(t+5)$ is derived according to equation 11.

$$x_{18j,k}(t+5) = a_{18j,k}(t)\ (x_{18j,k}(t) - h_{18j,k}(t)) + u_{16j,k}(t)\ (x_{16j,k}(t) - h_{16j,k}(t)) \tag{11}$$

Coefficients $a_{18j,k}$ and $u_{16j,k}$ are probabilities for surviving residual trees at time period t remaining in the 18-inch d.b.h. class or growing up from the 16-inch class, respectively, during the growth period t to $t+5$. Equation 11 appears to be a linear operation, but transition probabilities are functions of $\mathbf{x}(t)$ and $\mathbf{h}(t)$ in terms of initial and residual basal area as shown in equations 12 and 13. Similar to the first stage of FIBER, transition probabilities for each diameter class i, each species group j, and each stand k are estimated by equations 12 and 13, plus ingrowth by equation 14, where β is a vector of regression coefficients estimated from growth data. There are 13 sets of regression coefficients in FIBER, one set for each species group.

$$\begin{aligned} a_{i,j,k}(t) = {} & \beta_{0,aj} + \beta_{1,aj}IBA_k(t) + \beta_{2,aj}RBA_k(t) + \beta_{3,aj}D_i + \beta_{4,aj}PH_k(t) \\ & + \beta_{5,aj}D_i^2 + \beta_{6,aj}RBA_k^2(t) \end{aligned} \tag{12}$$

$$u_{i,j,k}(t) = \beta_{0,u_j} + \beta_{1,u_j} IBA_k(t) + \beta_{2,u_j} RBA_k(t) + \beta_{3,u_j} D_i + \beta_{4,u_j} PH_k(t) + \beta_{5,u_j} D_i^2 + \beta_{6,u_j} RBA_k^2(t) \tag{13}$$

$$f_{1,j,k} = \beta_{0,f_j} + \beta_{1,f_j} RBA_k(t) + \beta_{2,f_j} PH_k(t) + \beta_{3,f_j} PS_{j,k}(t) \tag{14}$$

Measures of stand stocking (RBA_k, IBA_k, PH_k, and PS_k) are functions of $\mathbf{x}(t)$ and $\mathbf{h}(t)$, computed as inner products of basal area at the midpoint of each d.b.h. class b_i and vectors $\mathbf{x}(t)$ and $\mathbf{h}(t)$ as shown in equations 15-18. Diameter classes range from 6 inches d.b.h. to *MD*, although 30 inches is the maximum recommended d.b.h. for the FIBER growth model. In equations 15 and 16, $IBA_k(t)$ and $RBA_k(t)$ include all species in a particular stand *k*, thus the subscript *j* was dropped for simplicity. In equation 17, *hd* represents hardwood species.

$$IBA_k(t) = 0.196\ x_{6,k}(t) + 0.349\ x_{8,k}(t) + \ldots + b_{MD}\ x_{MD,k}(t) \tag{15}$$

$$RBA_k(t) = 0.196\ (x_{6,k}(t) - h_{6,k}(t)) + \ldots + b_{MD,k}\ (x_{MD,k}(t) - h_{MD,k}(t)) \tag{16}$$

$$PH_k(t) = \frac{0.196\ x_{6,hd,k}(t) + 0.349\ x_{8,hd,k}(t) + \ldots + b_{MD}\ x_{MD,hd,k}(t)}{RBA_k(t)} \tag{17}$$

$$PS_{j,k}(t) = \frac{0.196\ x_{6,j,k}(t) + 0.349\ x_{8,j,k}(t) + \ldots + b_{MD}\ x_{MD,j,k}(t)}{RBA_k(t)} \tag{18}$$

In formulating explicit growth constraints, equations 15-18 are substituted into equations 12-14, and then for each diameter class, equations 12-14 are substituted into a general form of equation 11. A complete generalized growth constraint defined in equation 19 is constructed for each size class *i*, species group *j*, in each stand *k*. Basal area measured at the midpoint of each d.b.h. class is represented by b_i. Ingrowth defined in equation 14 is added to the smallest d.b.h. class in each time period *t*. Note that much of the nonlinearity associated with the model is attributed to squared terms involving decision variables in the growth constraints.

In practice, equation 19 must be expanded for each species group *j* and stand *k* included in the analyses. In the general form presented in equation 19, each initial stand variable $\mathbf{x}(t)$ and harvest control variable $\mathbf{h}(t)$ is indexed only by *i* for d.b.h. class, which is appropriate in the special case where there is only one stand containing a single species. In more practical applications, the NLP would contain growth constraints to represent all diameter classes, species groups, and stands included in the problem. FIBER can accommodate combinations of 13 species groups (Solomon, Hosmer, and Hayslett 1987).

$$
\begin{aligned}
x_i(t+1) = {} & (\beta_{0,a} + \beta_{1,a}\,[b_6 x_6(t) + \ldots + b_{MD}\,x_{MD}(t)] \\
& + \beta_{2,a}[b_6(x_6(t)-h_6(t)) + \ldots + b_{MD}(x_{MD}(t)-h_{MD}(t))] + \beta_{3,a}D_i \\
& + \beta_{4,a}[b_6 x_{6,hd}(t) + \ldots + b_{MD}x_{MD,hd}(t)] \;/\; [b_6(x_6(t)-h_6(t)) + \ldots + b_{md}(x_{MD}(t)-h_{MD}(t))] \\
& + \beta_{5,a}D_i^2 \\
& + \beta_{6,a}[b_6(x_6(t)-h_6(t)) + \ldots + b_{MD}(x_{MD}(t)-h_{MD}(t))]^2\ )\ (x_i(t)-h_i(t)) \\
& + (\ \beta_{0,u} + \beta_{1,u}\,[b_6 x_6(t) + \ldots + b_{MD}x_{MD}(t)] \\
& + \beta_{2,u}\,[b_6(x_6(t)-h_6(t)) + \ldots + b_{MD}(x_{MD}(t)-h_{MD}(t))] + \beta_{3,u}D_{i-1} \\
& + \beta_{4,u}[b_6 x_{6,hd}(t) + \ldots + b_{MD}x_{MD,hd}(t)] \;/\; [b_6(x_6(t)-h_6(t)) + \ldots + b_{MD}(x_{MD}(t)-h_{MD}(t))] \\
& + \beta_{5,u}D_{i-1}^{\ 2} \\
& + \beta_{6,u}\,[b_6(x_6(t)-h_6(t)) + \ldots + b_{MD}(x_{MD}(t)-h_{MD}(t))]^2\ )\ (x_{i-1}(t)-h_{i-1}(t))
\end{aligned}
\tag{19}
$$

EXAMPLE PROBLEM

The general management model was formulated as a sizable nonlinear programming problem and coded in the General Algebraic Modeling System (GAMS), a FORTRAN-based equation generator capable of representing complex models in compact form (Brooke, Kendrick, and Meeraus 1988). For problems with nonlinear constraints, the GAMS/MINOS optimizer employs a projected Lagrangian algorithm (Robinson 1972). An example problem is presented to demonstrate the performance of growth equations adapted from FIBER (Solomon, Hosmer, and Hayslett 1987).

The example problem simply required the NLP model to project individual stand growth to allow comparisons with actual stand growth and predicted stand growth using the FIBER simulator itself. Data were taken from a 55-year-old, second growth mixed hardwood stand on site index 64 (base age 50) for northern red oak. A 100 percent inventory from this 12.5-acre control stand was taken in 1964 and again in 1974 and 1984, allowing for comparison with 10- and 20-year growth projections from both FIBER and the NLP model.

The NLP model contained the objective function (equation 1), feasibility constraints (equations 4-6), growth constraints (equation 19), stand constraints to define the initial stand structure (equation 20), and management constraints to allow no periodic harvests (equation 21). Equation 20 simply requires that the initial stand structure $\mathbf{x}(0)$ must equal the 2-inch stand structure from the example stand $\mathbf{x}_0$ at time $t=0$.

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{20}$$

In order to project stand growth in the absence of harvests, the harvest control variable $\mathbf{h}(t)$ was constrained to be equal to zero for all time periods in equation 21.

$$\mathbf{h}(t) = 0 \tag{21}$$

With these constraints in place, the NLP problem simply determines the values for $\mathbf{x}(t)$, the initial number of trees in each species and size class, for each time period from $t=0$ to $t=20$ that satisfy the growth constraints derived from FIBER. Because no harvests are allowed in this example, the feasible region is reduced to a single point. The effect of the objective function has been nullified. Growth of the initial stand is deterministic, and should be approximately equal to projections from the FIBER software that utilizes the same regression coefficients.

A comparison of actual and projected stand structure and basal area stocking is shown in Table 1. Basal area projections from the nonlinear programming (NLP) formulation varied less than 2 percent from direct FIBER projections. Projected stand development varied from observed stand development less than 6 percent at 10 years and less than 8 percent at 20 years (Table 1). Direct FIBER projections were obtained from a modified algorithm that provided a 2-inch stand structure needed for this comparison (Marquis 1990). Discrepancies in stand structure projections are due to conversions from 1-inch stand structures generated by FIBER to 2-inch stand structures. Results indicated that modeling the growth simulator as a component of a nonlinear programming problem provided adequate stand table projections to allow numerical evaluations of management strategies.

Table 1.—Stand growth projections for a 55-year-old mixed hardwood stand on site index 64

		10-year			20-year		
Dbh	**Initial**	**Actual**	**NLP**	**FIBER**	**Actual**	**NLP**	**FIBER**
		number of stems/acre					
6	77.1	57.1	52.7	42.7	45.5	40.6	30.6
8	38.6	40.1	50.0	53.5	31.5	45.8	45.9
10	20.7	25.3	31.2	30.5	25.7	37.7	41.2
12	0.2	14.3	16.7	16.5	17.8	24.0	23.4
14	7.7	9.1	9.8	9.4	13.0	13.9	12.8
16	6.1	7.6	7.1	7.0	9.0	8.7	8.1
18	4.6	4.9	5.4	5.8	7.0	6.3	6.2
20	3.1	3.3	3.9	3.9	3.6	4.6	4.6
22	1.8	2.7	2.4	2.4	3.0	3.0	3.0
24	1.7	1.7	1.7	1.7	2.0	1.9	1.9
26	1.4	1.1	1.4	1.4	1.7	1.4	1.4
208	0	1.0	0.4	0.4	1.1	0.6	0.6
		ft^2/acre					
Basal area	96.8	108.6	115.1	113.8	123.7	133.2	130.5

DISCUSSION

Formulating available stand-growth models for use in mathematical programming applications allows for relatively thorough analyses of management alternatives. Growth models typically are used to project the outcome of a specific cutting strategy. Users may compare projections for a range of strategies in an effort to find a "best" course of action. A major drawback of this approach is that superior strategies may be omitted from the analyses and thus overlooked. The mathematical programming approach allows all feasible alternatives to be considered, even those with very slight mathematical variation from others.

With nonlinear growth constraints from FIBER (Solomon, Hosmer, and Hayslett 1987), it is possible to construct multi-stand, multi-resource management formulations for forests where growth projections are valid. This type of model can optimize even-aged or uneven-aged silviculture for individual stands using the appropriate combination of management constraints. In addition, constraints such as minimum woodflow from an aggregate of stands may be imposed to assess the impact forest-level objectives on optimum harvest strategies for individual stands within the multi-stand problem.

Problem dimensions can increase rapidly when adding size classes, species groups, and time periods in a multi-stand problem. Relatively large problems may be solved using GAMS/386 for IBM-compatible personal computers equipped with appropriate memory. The model presented in this report was expanded to 12 size classes, 2 species groups, 3 stands, and 21 time periods covering a 100-year planning horizon. Optimal solutions were obtained in less than 3 hours cpu time on a 386-20 MHz processor.

LITERATURE CITED

Adams, D.M.; A.R. Ek. 1974. Optimizing the management of uneven-aged forest stands. Canadian Journal of Forest Research 4(3):274-287.

Brooke, A.; D. Kendrick; A. Meeraus. 1988. GAMS: A user's guide. San Francisco, CA: The Scientific Press. 289 p.

Buongiorno, J.; B.R. Michie. 1980. A matrix model of uneven-aged forest management. Forest Science 26(4):609-625.

Getz, W.M.; R.G. Haight. 1989. Population harvesting demographic models of fish, forest, and animal resources. Princeton, NJ: Princeton University Press. 359 p.

Haight, R.G. 1987. Evaluating the efficiency of even-aged and uneven-aged stand management. Forest Science 33(1):116-134.

Hilt, D.E. 1985. OAKSIM: An individual tree growth and yield simulator for managed, even-aged, upland oak stands. Res. Pap. NE-562. Radnor, PA: U.S. Department of Agriculture, Forest Service. Northeastern Forest Experiment Station. 21 p.

Marquis, D.A. 1990. A multi-resource silvicultural decision model for forests of the Northeastern United States. In: Proceedings, Division I, IUFRO Congress, Montreal, Canada. p. 419-431.

Marquis, D.A.; R.L. Ernst. 1992. User's guide to SILVAH: stand analysis, prescription, and management simulator program for hardwood stands of the Alleghenies. Gen. Tech. Rep. NE-162. Radnor, PA: U.S. Department of Agriculture, Forest Service, Northeastern Forest Experiment Station. 124 p.

Martin, G.L. 1982. Investment-efficient stocking guides for all-aged northern hardwood forests. Res. Pap. R3129. Madison, WI: University of Wisconsin, College of Agriculture and Life Science. 12 p.

Robinson, R. A. 1972. A quadratically convergent algorithm for general nonlinear programming problems. Mathematical Programming 3. 145-156.

Solomon, D.S.; R.A. Hosmer; H.T. Hayslett, Jr. 1987. FIBER handbook: A growth model for spruce-fir and northern hardwood types. Res. Pap. NE-602. Broomall, PA: U.S. Department of Agriculture, Forest Service, Northeastern Forest Experiment Station. 19 p.

Teck, R.M. 1990. NE-TWIGS 3.0: An individual tree growth and yield projection system for the Northeastern United States. The Compiler. 8(1):25-27.